CSEP 590
Data Compression
Autumn 2007

Predictive Coding
Burrows-Wheeler Transform

Predictive Coding

- The next symbol can be statistically predicted from the past.
  - Code with context
  - Code the difference
  - Move to front, then code
- Goal of prediction
  - The prediction should make the distribution of probabilities of the next symbol as skewed as possible
  - After prediction there is no way to predict more so we are in the first order entropy model

Bad and Good Prediction

- From information theory – The lower the information the fewer bits are needed to code the symbol.

$$\inf(a) = \log_2\left(\frac{1}{P(a)}\right)$$

- Examples:
  - P(a) = 1023/1024, inf(a) = .000977
  - P(a) = 1/2, inf(a) = 1
  - P(a) = 1/1024, inf(a) = 10

Entropy

- Entropy is the expected number of bits to code a symbol in the model with a_i having probability $P(a_i)$.

$$H = \sum_{i=1}^{m} P(a_i) \log_2 \left(\frac{1}{P(a_i)} \right)$$

- Good coders should be close to this bound.
  - Arithmetic
  - Huffman
  - Golomb
  - Tunstall

PPM

- Prediction with Partial Matching
  - Cleary and Witten (1984)
  - Tries to find a good context to code the next symbol

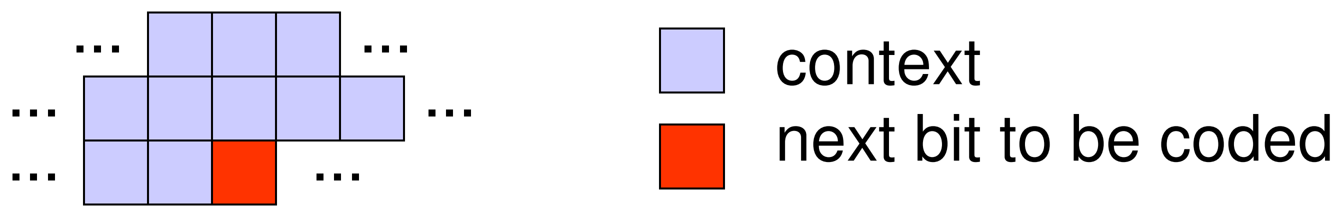
good → he

context	a	e	i	r	s	y
the	0	0	5	7	4	7
he	10	1	7	10	9	7
e	12	2	10	15	10	10
<nil>	50	70	30	35	40	13

- Uses adaptive arithmetic coding for each context

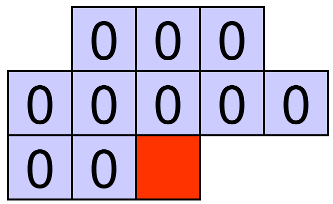
JBIG

- Coder for binary images
  – documents
  – graphics
- Codes in scan line order using context from the same and previous scan lines.



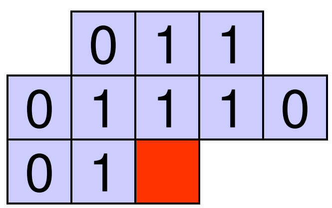
- Uses adaptive arithmetic coding with context

JBIG Example



next bit	0	1
frequency	100	10

$$H = \frac{10}{110}\log(\frac{110}{10}) + \frac{100}{110}\log(\frac{110}{100}) = .44$$



next bit	0	1
frequency	15	50

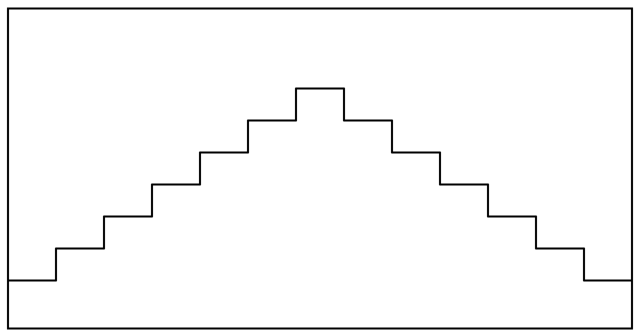
$$H = \frac{15}{65}\log(\frac{65}{15}) + \frac{50}{65}\log(\frac{65}{50}) = .78$$

Issues with Context

- Context dilution
  – If there are too many contexts then too few symbols are coded in each context, making them ineffective because of the zero-frequency problem.
- Context saturation
  – If there are too few contexts then the contexts might not be as good as having more contexts.
- Wrong context
  – Again poor predictors.

Prediction by Differencing

- Used for Numerical Data
- Example: 2 3 4 5 6 7 8 7 6 5 4 3 2



- Transform to 2 1 1 1 1 1 1 –1 –1 –1 –1 –1 –1
  - much lower first-order entropy

General Differencing

- Let $x_1, x_2, \ldots, x_n$ be some numerical data that is correlated, that is x_i is near x_{i+1}
- Better compression can result from coding
  $x_1, x_2 - x_1, x_3 - x_2, \ldots, x_n - x_{n-1}$
- This idea is used in
  - signal coding
  - audio coding
  - video coding
- There are fancier prediction methods based on linear combinations of previous data, but these may require training.

Move to Front Coding

- Non-numerical data
- The data have a relatively small working set that changes over the sequence.
- Example: a b a b a a b c c b b c c c c b d b c c
- Move to Front algorithm
  - Symbols are kept in a list indexed 0 to m-1
  - To code a symbol output its index and move the symbol to the front of the list

Example

- Example: a b a b a a b c c b b c c c c b d b c c
  0

0	1	2	3
a	b	c	d

Example

- Example: a b a b a a b c c b b c c c c b d b c c
  0 1

0	1	2	3
a	b	c	d

↓

0	1	2	3
b	a	c	d

Example

- Example: a b a b a a b c c b b c c c c b d b c c
  0 1 1

0	1	2	3
b	a	c	d

↓

0	1	2	3
a	b	c	d

Example

- Example: a b a b a a b c c b b c c c c b d b c c
  0 1 1 1

0	1	2	3
a	b	c	d

↓

0	1	2	3
b	a	c	d

Example

- Example: a b a b a a b c c b b c c c c b d b c c
  0 1 1 1 1

0	1	2	3
b	a	c	d

↓

0	1	2	3
a	b	c	d

Example

- Example: <u>a</u> <u>b</u> <u>a</u> <u>b</u> <u>a</u> <u>a</u> b c c b b c c c c b d b c c
  0 1 1 1 1 0

0	1	2	3
a	b	c	d

Example

- Example: a b a b a a b c c b b c c c c b d b c c
  0 1 1 1 1 0 1

0	1	2	3
a	b	c	d

↓

0	1	2	3
b	a	c	d

Example

- Example: a b a b a a b c c b b c c c c b d b c c
  0 1 1 1 1 0 1 2

0	1	2	3
b	a	c	d

↓

0	1	2	3
c	b	a	d

Example

- Example: a b a b a a b c c b b c c c c b d b c c
  0 1 1 1 1 0 1 2 0 1 0 1 0 00 1 3 1 2 0

0	1	2	3
c	b	d	a

Example

- Example: a b a b a a b c c b b c c c c b d b c c
  0 1 1 1 1 0 1 2 0 1 0 1 0 00 1 3 1 2 0

Frequencies of {a, b, c, d}

a	b	c	d
4	7	8	1

Frequencies of {0, 1, 2, 3}

0	1	2	3
8	9	2	1

Extreme Example

Input:

aaaaaaaaaabbbbbbbbbbccccccccccdddddddddd

Output

0000000000100000000020000000003000000000

Frequencies of a b c d

a	b	c	d
10	10	10	10

Frequencies of 0 1 2 3

0	1	2	3
37	1	1	1

Burrows-Wheeler Transform

- Burrows-Wheeler, 1994
- BW Transform creates a representation of the data which has a small working set.
- The transformed data is compressed with move to front compression.
- The decoder is quite different from the encoder.
- The algorithm requires processing the entire string at once (it is not on-line).
- It is a remarkably good compression method.

Encoding Example

- abracadabra

1. Create all cyclic shifts of the string.

```
0   abracadabra
1   bracadabraa
2   racadabraab
3   acadabraabr
4   cadabraabra
5   adabraabrac
6   dabraabraca
7   abraabracad
8   braabracada
9   raabracadab
10  aabracadabr
```

Encoding Example

2. Sort the strings alphabetically in to array A

0	abracadabra
1	bracadabraa
2	racadabraab
3	acadabraabr
4	cadabraabra
5	adabraabrac
6	dabraabraca
7	abraabracad
8	braabracada
9	raabracadab
10	aabracadabr

→

A

0	aabracadabr
1	abraabracad
2	abracadabra
3	acadabraabr
4	adabraabrac
5	braabracada
6	bracadabraa
7	cadabraabra
8	dabraabraca
9	raabracadab
10	racadabraab

Encoding Example

3. L = the last column

```
A
  0  aabracadabr
  1  abraabracad
  2  abracadabra
  3  acadabraabr
  4  adabraabrac
  5  braabracada
  6  bracadabraa
  7  cadabraabra
  8  dabraabraca
  9  raabracadab
  10 racadabraab
```

L = rdarcaaaabb

Encoding Example

4. Transmit X the index of the input in A and L (using move to front coding).

A

```
0   aabracadabr
1   abraabracad
2   abracadabra
3   acadabraabr
4   adabraabrac
5   braabracada
6   bracadabraa
7   cadabraabra
8   dabraabraca
9   raabracadab
10  racadabraab
```

```
L = rdarcaaaabb
X = 2
```

Why BW Works

- Ignore decoding for the moment.
- The prefix of each shifted string is a context for the last symbol.
  - The last symbol appears just before the prefix in the original.
- By sorting, similar contexts are adjacent.
  - This means that the predicted last symbols are similar.

Decoding Example

- We first decode assuming some information. We then show how to compute the information.
- Let A^s be A shifted by 1

A		A^s	
0	aabracadabr	0	raabracadab
1	abraabracad	1	dabraabraca
2	abracadabra	2	aabracadabr
3	acadabraabr	3	racadabraab
4	adabraabrac	4	cadabraabra
5	braabracada	5	abraabracad
6	bracadabraa	6	abracadabra
7	cadabraabra	7	acadabraabr
8	dabraabraca	8	adabraabrac
9	raabracadab	9	braabracada
10	racadabraab	10	bracadabraa

Decoding Example

- Assume we know the mapping T[i] is the index in A^s of the string i in A.
- T = [2 5 6 7 8 9 10 4 1 0 3]

A		A^s	
0	aabracadabr	0	raabracadab
1	abraabracad	1	dabraabraca
2	abracadabra	2	aabracadabr
3	acadabraabr	3	racadabraab
4	adabraabrac	4	cadabraabra
5	braabracada	5	abraabracad
6	bracadabraa	6	abracadabra
7	cadabraabra	7	acadabraabr
8	dabraabraca	8	adabraabrac
9	raabracadab	9	braabracada
10	racadabraab	10	bracadabraa

Decoding Example

- Let F be the first column of A, it is just L, sorted.

F =

0	1	2	3	4	5	6	7	8	9	10
a	a	a	a	a	b	b	c	d	r	r

T =

0	1	2	3	4	5	6	7	8	9	10
2	5	6	7	8	9	10	4	1	0	3

- Follow the pointers in T in F to recover the input starting with X.

Decoding Example

F =	0	1	2	3	4	5	6	7	8	9	10
	a	a	a	a	a	b	b	c	d	r	r

T =	0	1	2	3	4	5	6	7	8	9	10
	2	5	6	7	8	9	10	4	1	0	3

a

Decoding Example

F =	0	1	2	3	4	5	6	7	8	9	10
	a	a	a	a	a	b	b	c	d	r	r

T =	0	1	2	3	4	5	6	7	8	9	10
	2	5	6	7	8	9	10	4	1	0	3

ab

Decoding Example

F =

0	1	<u>2</u>	3	4	5	<u>6</u>	7	8	9	<u>10</u>
a	a	a	a	a	b	b	c	d	r	r

T =

0	1	<u>2</u>	3	4	5	<u>6</u>	7	8	9	<u>10</u>
2	5	6	7	8	9	10	4	1	0	3

abr

Decoding Example

- Why does this work?
- The first symbol of A[T[i]] is the second symbol of A[i] because $A^s[T[i]] = A[i]$.

A

		T
0	aabracadabr	2
1	abraabracad	5
2	abracadabra	6
3	acadabraabr	7
4	adabraabrac	8
5	braabracada	9
6	bracadabraa	10
7	cadabraabra	4
8	dabraabraca	1
9	raabracadab	0
10	racadabraab	3

A^s

0	raabracadab
1	dabraabraca
2	aabracadabr
3	racadabraab
4	cadabraabra
5	abraabracad
6	abracadabra
7	acadabraabr
8	adabraabrac
9	braabracada
10	bracadabraa

Decoding Example

- How do we compute F and T from L and X?

  F is just L sorted

	0	1	2	3	4	5	6	7	8	9	10
F =	a	a	a	a	a	b	b	c	d	r	r
L =	r	d	a	r	c	a	a	a	a	b	b

Note that L is the first column of A^s and A^s is in the same order as A.

If i is the k-th x in F then T[i] is the k-th x in L.

Decoding Example

	0	1	2	3	4	5	6	7	8	9	10
F =	a	a	a	a	a	b	b	c	d	r	r
L =	r	d	a	r	c	a	a	a	a	b	b

	0	1	2	3	4	5	6	7	8	9	10
T =	2	5	6	7	8						

Decoding Example

	0	1	2	3	4	5	6	7	8	9	10
F =	a	a	a	a	a	b	b	c	d	r	r
L =	r	d	a	r	c	a	a	a	a	b	b

T=	0	1	2	3	4	5	6	7	8	9	10
	2	5	6	7	8	9	10				

Decoding Example

	0	1	2	3	4	5	6	7	8	9	10
F =	a	a	a	a	a	b	b	c	d	r	r
L =	r	d	a	r	c	a	a	a	a	b	b

	0	1	2	3	4	5	6	7	8	9	10
T=	2	5	6	7	8	9	10	4			

Decoding Example

	0	1	2	3	4	5	6	7	8	9	10
F =	a	a	a	a	a	b	b	c	d	r	r

L =	r	d	a	r	c	a	a	a	a	b	b

	0	1	2	3	4	5	6	7	8	9	10
T=	2	5	6	7	8	9	10	4	1		

Decoding Example

	0	1	2	3	4	5	6	7	8	9	10
F =	a	a	a	a	a	b	b	c	d	r	r
L =	r	d	a	r	c	a	a	a	a	b	b

	0	1	2	3	4	5	6	7	8	9	10
T =	2	5	6	7	8	9	10	4	1	0	3

Notes on BW

- Alphabetic sorting does not need the entire cyclic shifted inputs.
  - Sort the indices of the string
  - Most significant symbols first radix sort works
- There are high quality practical implementations
  - Bzip
  - Bzip2 (seems to be w/o patents)

Encoding Exercise

Encode the string abababababababab = $(ab)^8$

1. Find L and X
2. Do move-to-front coding of L.
3. Estimate the length of the code using first order entropy.

Decoding Exercise

Decode L = baaaaaba, X = 6
1. First Compute F and T
2. Use those to decode.