Histogram Processing

Histogram Processing

- Histogram Equalization
- Histogram Matching
- Local Histogram Processing
- Using Histogram Statistics for Image Enhancement

Histogram Processing

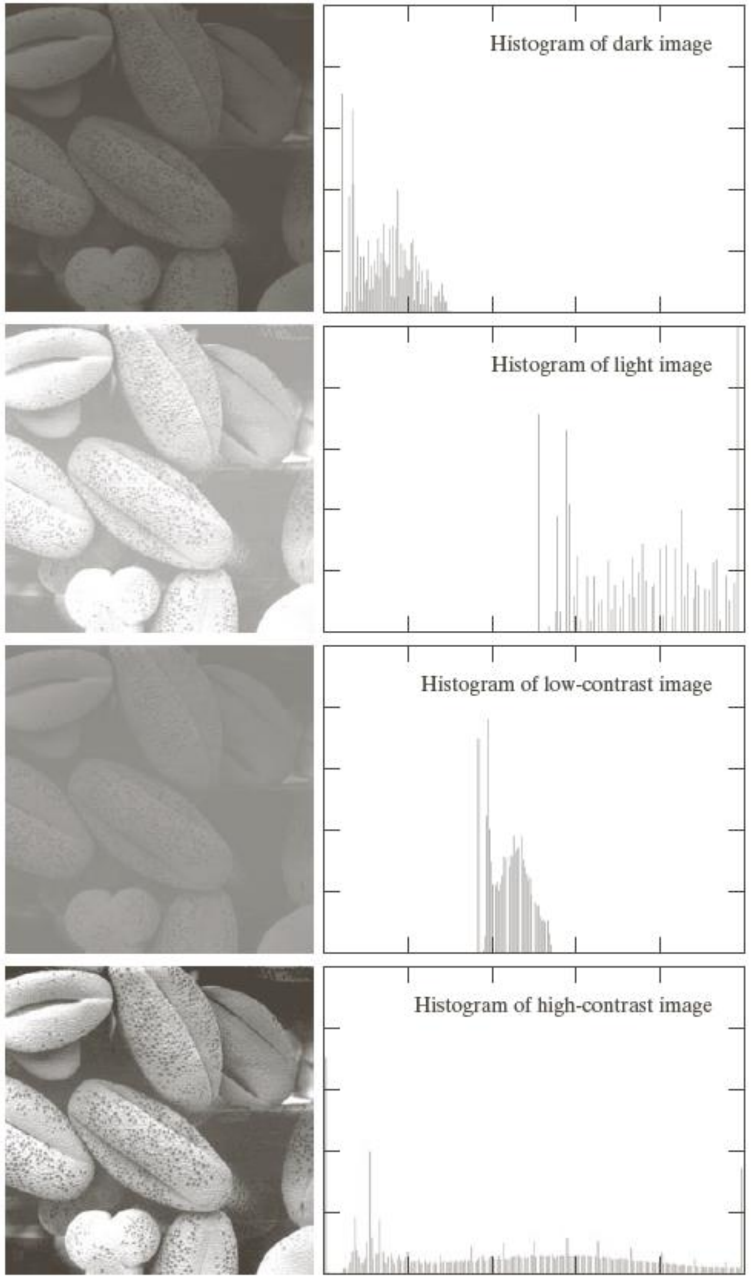
Histogram $h(r_k) = n_k$

r_k is the k^{th} intensity value

n_k is the number of pixels in the image with intensity r_k

Normalized histogram $p(r_k) = \frac{n_k}{MN}$

n_k: the number of pixels in the image of size $M \times N$ with intensity r_k


Histogram of dark image
Histogram of light image
Histogram of low-contrast image
Histogram of high-contrast image

Histogram Equalization

The intensity levels in an image may be viewed as random variables in the interval [0, L-1].

Let $p_r(r)$ and $p_s(s)$ denote the probability density function (PDF) of random variables r and s.

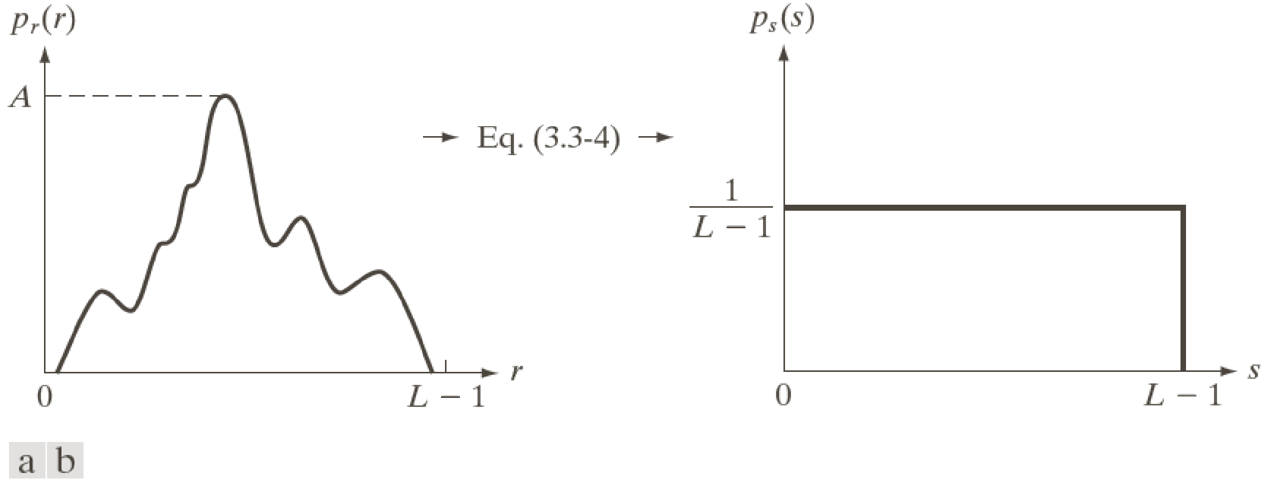


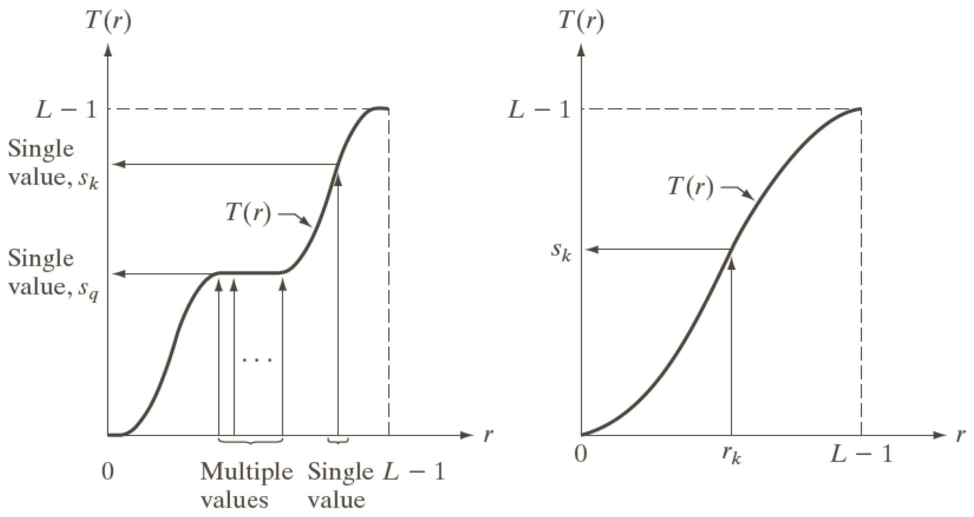
FIGURE 3.18 (a) An arbitrary PDF. (b) Result of applying the transformation in Eq. (3.3-4) to all intensity levels, r. The resulting intensities, s, have a uniform PDF, independently of the form of the PDF of the r's.

Histogram Equalization

$$s = T(r) \qquad 0 \le r \le L-1$$

a. T(r) is a strictly monotonically increasing function in the interval $0 \le r \le L\text{-}1$;

b. $0 \le T(r) \le L\text{-}1$ for $0 \le r \le L\text{-}1$.



a b

FIGURE 3.17 (a) Monotonically increasing function, showing how multiple values can map to a single value. (b) Strictly monotonically increasing function. This is a one-to-one mapping, both ways.

Histogram Equalization

$$s = T(r) \qquad 0 \le r \le L - 1$$

a. T(r) is a strictly monotonically increasing function in the interval $0 \le r \le L\text{-}1$;

b. $0 \le T(r) \le L\text{-}1$ for $0 \le r \le L\text{-}1$.

$T(r)$ is continuous and differentiable.

$$p_s(s)ds = p_r(r)dr$$

Histogram Equalization

$$s = T(r) = (L-1)\int_0^r p_r(w)dw$$

$$\frac{ds}{dr} = \frac{dT(r)}{dr} = (L-1)\frac{d}{dr}\left[\int_0^r p_r(w)dw\right]$$

$$= (L-1)p_r(r)$$

$$p_s(s) = \frac{p_r(r)dr}{ds} = p_r(r) \Big/ \left(\frac{ds}{dr}\right) = p_r(r) \Big/ \left((L-1)p_r(r)\right) = \frac{1}{L-1}$$

Example

Suppose that the (continuous) intensity values in an image have the PDF

$$p_r(r) = \begin{cases} \dfrac{2r}{(L-1)^2}, & \text{for } 0 \le r \le L\text{-}1 \\ 0, & \text{otherwise} \end{cases}$$

Find the transformation function for equalizing the image histogram.

Example

$$s = T(r) = (L-1)\int_0^r p_r(w)dw$$

$$= (L-1)\int_0^r \frac{2w}{(L-1)^2}dw$$

$$= \frac{r^2}{L-1}$$

Histogram Equalization

Continuous case:

$$s = T(r) = (L-1)\int_0^r p_r(w)\,dw$$

Discrete values:

$$s_k = T(r_k) = (L-1)\sum_{j=0}^{k} p_r(r_j)$$

$$= (L-1)\sum_{j=0}^{k} \frac{n_j}{MN} = \frac{L-1}{MN}\sum_{j=0}^{k} n_j \qquad \text{k=0,1,..., L-1}$$

Example: Histogram Equalization

Suppose that a 3-bit image (L=8) of size 64 × 64 pixels (MN = 4096) has the intensity distribution shown in following table.
Get the histogram equalization transformation function and give the $p_s(s_k)$ for each s_k.

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

Example: Histogram Equalization

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

$$s_0 = T(r_0) = 7\sum_{j=0}^{0} p_r(r_j) = 7 \times 0.19 = 1.33 \quad \rightarrow 1$$

$$s_1 = T(r_1) = 7\sum_{j=0}^{1} p_r(r_j) = 7 \times (0.19 + 0.25) = 3.08 \quad \rightarrow 3$$

$$s_2 = 4.55 \quad \rightarrow 5 \qquad s_3 = 5.67 \quad \rightarrow 6$$

$$s_4 = 6.23 \quad \rightarrow 6 \qquad s_5 = 6.65 \quad \rightarrow 7$$

$$s_6 = 6.86 \quad \rightarrow 7 \qquad s_7 = 7.00 \quad \rightarrow 7$$

Example: Histogram Equalization

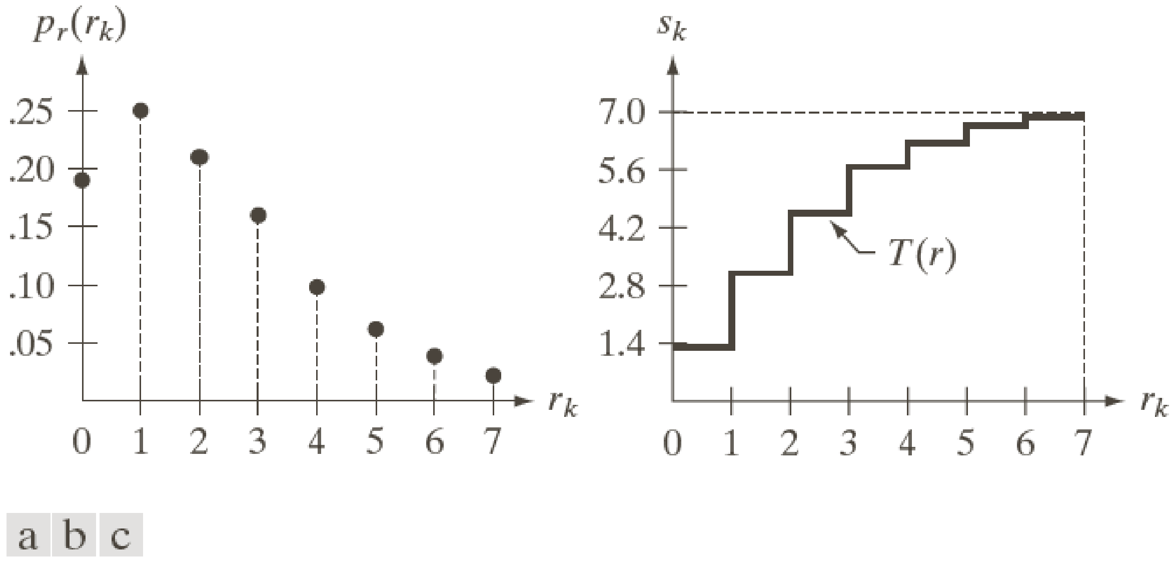


FIGURE 3.19 Illustration of histogram equalization of a 3-bit (8 intensity levels) image. (a) Original histogram. (b) Transformation function. (c) Equalized histogram.

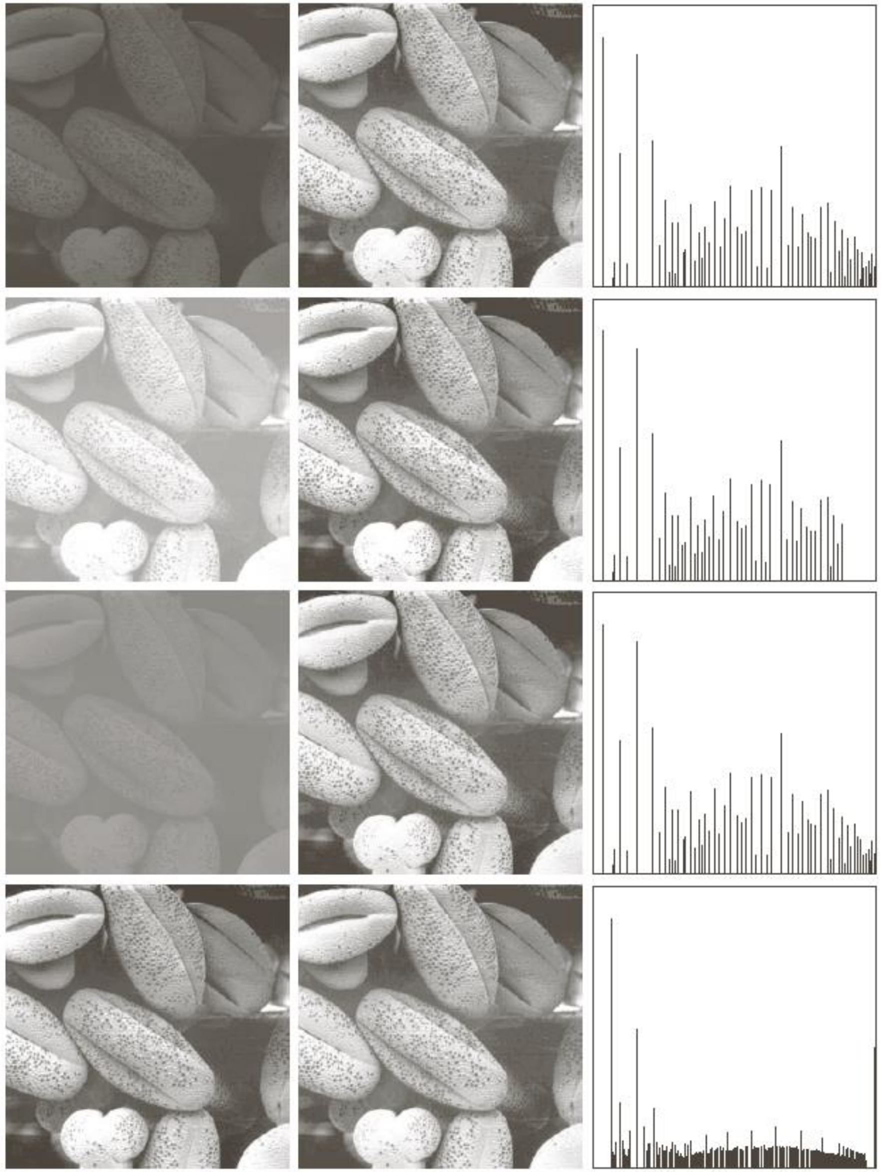


FIGURE 3.20 Left column: images from Fig. 3.16. Center column: corresponding histogram-equalized images. Right column: histograms of the images in the center column.

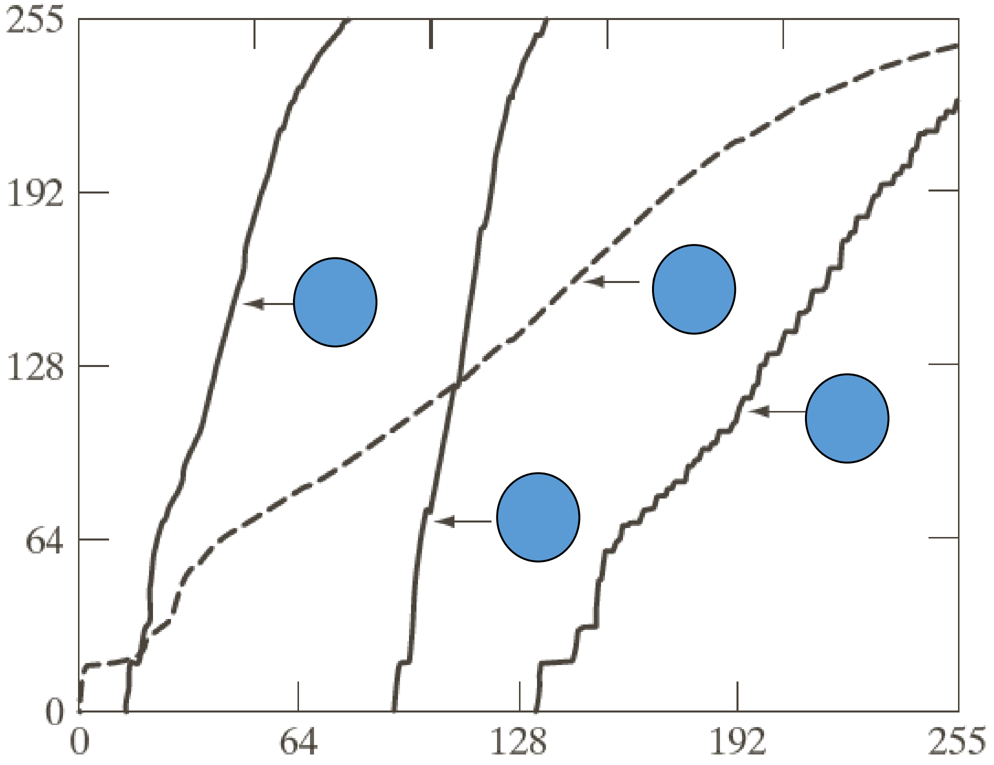


FIGURE 3.21 Transformation functions for histogram equalization. Transformations (1) through (4) were obtained from the histograms of the images (from top to bottom) in the left column of Fig. 3.20 using Eq. (3.3-8).

Histogram Matching

Histogram matching (histogram specification)

— generate a processed image that has a specified histogram

Let $p_r(r)$ and $p_z(z)$ denote the continous probability density functions of the variables r and z. $p_z(z)$ is the specified probability density function.

Let s be the random variable with the probability

$$s = T(r) = (L-1)\int_0^r p_r(w)dw$$

Define a random variable z with the probability

$$G(z) = (L-1)\int_0^z p_z(t)dt = s$$

Histogram Matching

$$s = T(r) = (L-1)\int_0^r p_r(w)dw$$

$$G(z) = (L-1)\int_0^z p_z(t)dt = s$$

$$z = G^{-1}(s) = G^{-1}\left[T(r)\right]$$

Histogram Matching: Procedure

- Obtain $p_r(r)$ from the input image and then obtain the values of s

$$s = (L-1)\int_0^r p_r(w)dw$$

- Use the specified PDF and obtain the transformation function G(z)

$$G(z) = (L-1)\int_0^z p_z(t)dt = s$$

- Mapping from s to z

$$z = G^{-1}(s)$$

Histogram Matching: Example

Assuming continuous intensity values, suppose that an image has the intensity PDF

$$p_r(r) = \begin{cases} \dfrac{2r}{(L-1)^2}, & \text{for } 0 \le r \le L\text{-}1 \\ 0\ , & \text{otherwise} \end{cases}$$

Find the transformation function that will produce an image whose intensity PDF is

$$p_z(z) = \begin{cases} \dfrac{3z^2}{(L-1)^3}, & \text{for } 0 \le z \le (L\text{-}1) \\ 0, & \text{otherwise} \end{cases}$$

Histogram Matching: Example

Find the histogram equalization transformation for the input image

$$s = T(r) = (L-1)\int_0^r p_r(w)dw = (L-1)\int_0^r \frac{2w}{(L-1)^2}dw = \frac{r^2}{L-1}$$

Find the histogram equalization transformation for the specified histogram

$$G(z) = (L-1)\int_0^z p_z(t)dt = (L-1)\int_0^z \frac{3t^2}{(L-1)^3}dt = \frac{z^3}{(L-1)^2} = s$$

The transformation function

$$z = \left[(L-1)^2 s\right]^{1/3} = \left[(L-1)^2 \frac{r^2}{L-1}\right]^{1/3} = \left[(L-1)r^2\right]^{1/3}$$

Histogram Matching: Discrete Cases

- Obtain $p_r(r_j)$ from the input image and then obtain the values of s_k, round the value to the integer range [0, L-1].

$$s_k = T(r_k) = (L-1)\sum_{j=0}^{k} p_r(r_j) = \frac{(L-1)}{MN}\sum_{j=0}^{k} n_j$$

- Use the specified PDF and obtain the transformation function $G(z_q)$, round the value to the integer range [0, L-1].

$$G(z_q) = (L-1)\sum_{i=0}^{q} p_z(z_i) = s_k$$

- Mapping from s_k to z_q

$$z_q = G^{-1}(s_k)$$

Example: Histogram Matching

Suppose that a 3-bit image (L=8) of size 64 × 64 pixels (MN = 4096) has the intensity distribution shown in the following table (on the left). Get the histogram transformation function and make the output image with the specified histogram, listed in the table on the right.

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

z_q	Specified $p_z(z_q)$
$z_0 = 0$	0.00
$z_1 = 1$	0.00
$z_2 = 2$	0.00
$z_3 = 3$	0.15
$z_4 = 4$	0.20
$z_5 = 5$	0.30
$z_6 = 6$	0.20
$z_7 = 7$	0.15

Example: Histogram Matching

Obtain the scaled histogram-equalized values,

$$s_0 = 1, s_1 = 3, s_2 = 5, s_3 = 6, s_4 = 7,$$
$$s_5 = 7, s_6 = 7, s_7 = 7.$$

Compute all the values of the transformation function G,

$$G(z_0) = 7\sum_{j=0}^{0} p_z(z_j) = 0.00 \rightarrow 0$$

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

$$G(z_1) = 0.00 \rightarrow 0 \qquad G(z_2) = 0.00 \rightarrow 0$$
$$G(z_3) = 1.05 \rightarrow 1 \qquad G(z_4) = 2.45 \rightarrow 2$$
$$G(z_5) = 4.55 \rightarrow 5 \qquad G(z_6) = 5.95 \rightarrow 6$$
$$G(z_7) = 7.00 \rightarrow 7$$

z_q	Specified $p_z(z_q)$	Actual $p_z(z_k)$
$z_0 = 0$	0.00	0.00
$z_1 = 1$	0.00	0.00
$z_2 = 2$	0.00	0.00
$z_3 = 3$	0.15	0.19
$z_4 = 4$	0.20	0.25
$z_5 = 5$	0.30	0.21
$z_6 = 6$	0.20	0.24
$z_7 = 7$	0.15	0.11

Example: Histogram Matching

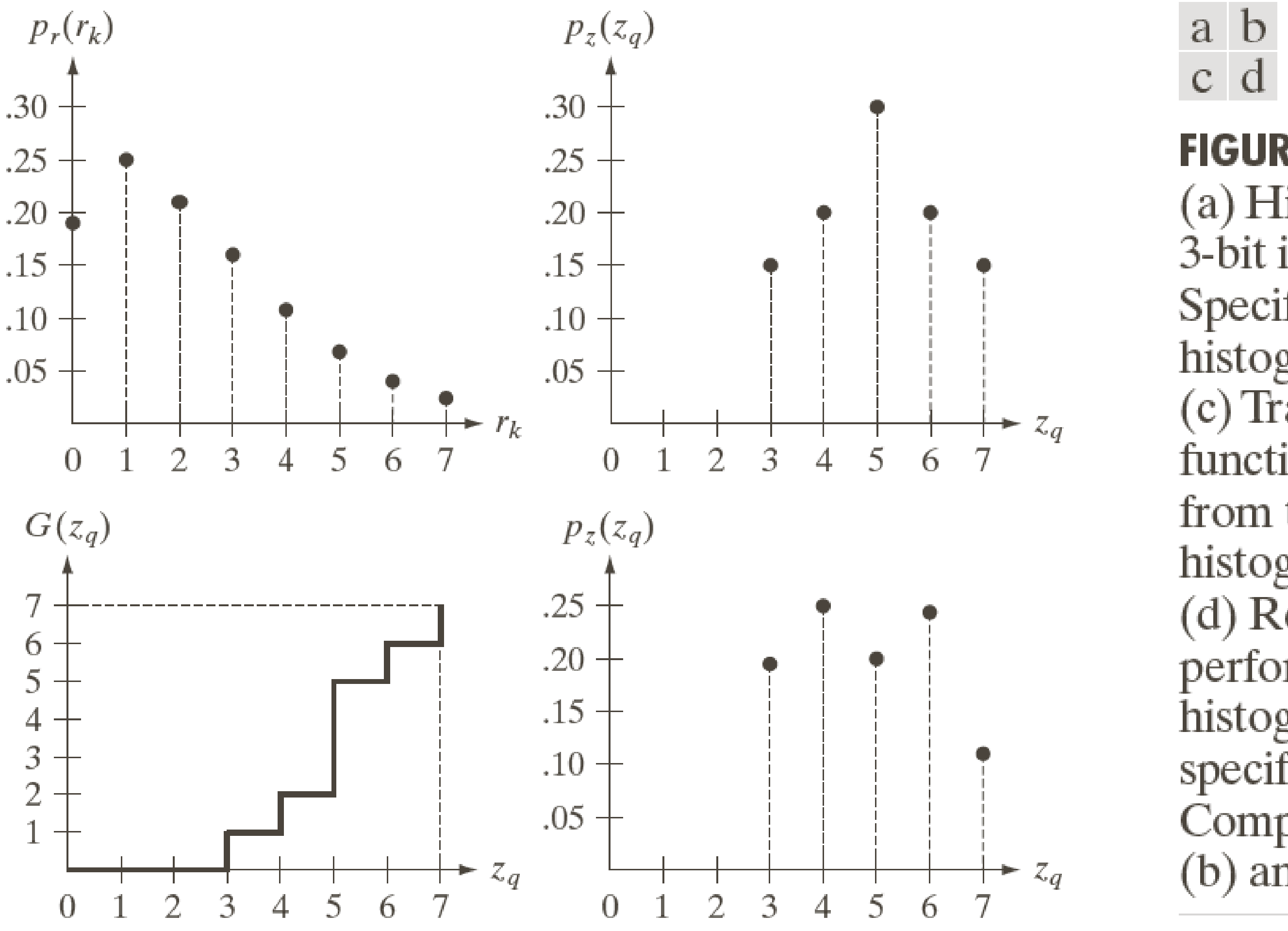


FIGURE 3.22
(a) Histogram of a 3-bit image. (b) Specified histogram. (c) Transformation function obtained from the specified histogram. (d) Result of performing histogram specification. Compare (b) and (d).

Example: Histogram Matching

Obtain the scaled histogram-equalized values,

$$s_0 = 1, s_1 = 3, s_2 = 5, s_3 = 6, s_4 = 7,$$
$$s_5 = 7, s_6 = 7, s_7 = 7.$$

Compute all the values of the transformation function G,

$$G(z_0) = 7\sum_{j=0}^{0} p_z(z_j) = 0.00 \rightarrow 0$$

$G(z_1) = 0.00 \rightarrow 0$ $G(z_2) = 0.00 \rightarrow 0$

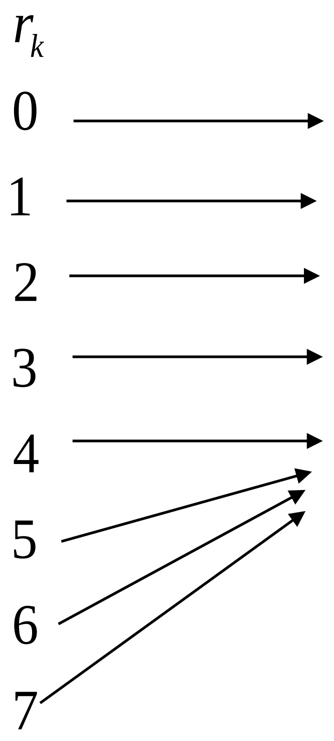
$G(z_3) = 1.05 \rightarrow 1$ **s_0** $G(z_4) = 2.45 \rightarrow 2$ **s_1**

$G(z_5) = 4.55 \rightarrow 5$ **s_2** $G(z_6) = 5.95 \rightarrow 6$ **s_3**

$G(z_7) = 7.00 \rightarrow 7$ **s_4 s_5 s_6 s_7**

Example: Histogram Matching

$$s_0 = 1, s_1 = 3, s_2 = 5, s_3 = 6, s_4 = 7,$$
$$s_5 = 7, s_6 = 7, s_7 = 7.$$



s_k	$\rightarrow$	z_q
1	$\rightarrow$	3
3	$\rightarrow$	4
5	$\rightarrow$	5
6	$\rightarrow$	6
7	$\rightarrow$	7

Example: Histogram Matching

$$r_k \rightarrow z_q$$

$$0 \rightarrow 3$$

$$1 \rightarrow 4$$

$$2 \rightarrow 5$$

$$3 \rightarrow 6$$

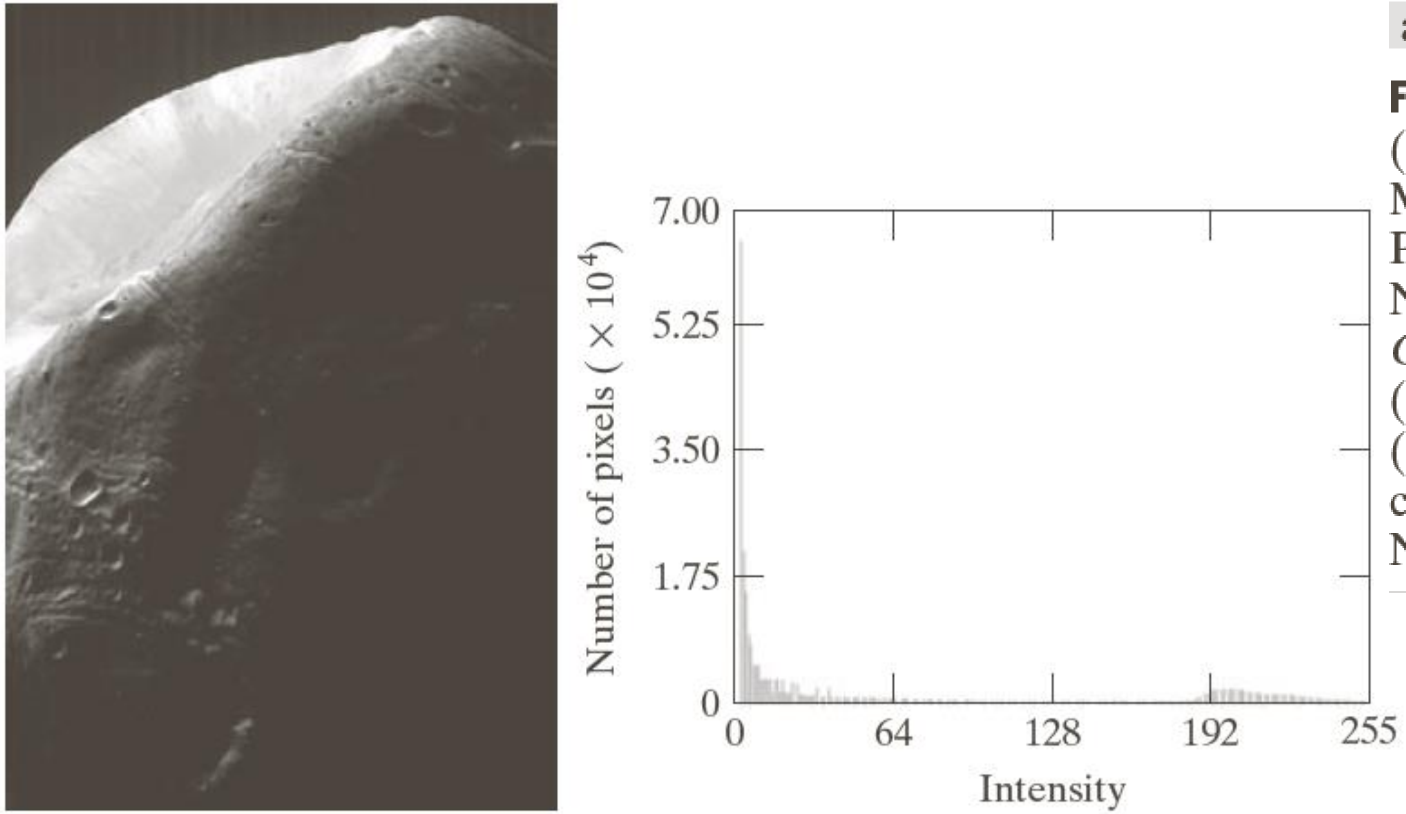
$$4 \rightarrow 7$$

$$5 \rightarrow 7$$

$$6 \rightarrow 7$$

$$7 \rightarrow 7$$

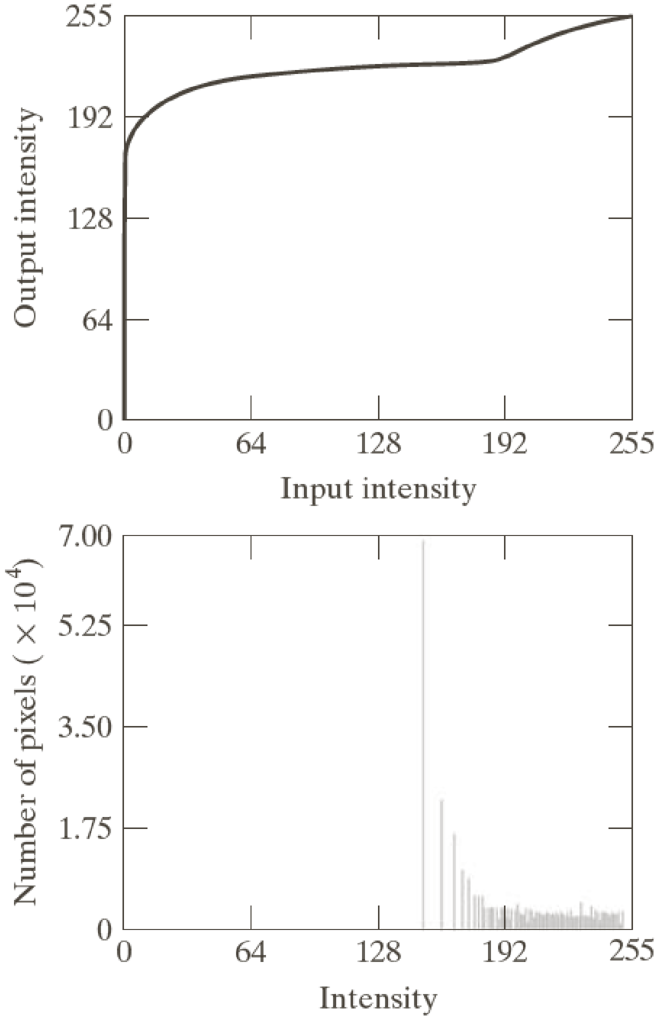
Example: Histogram Matching

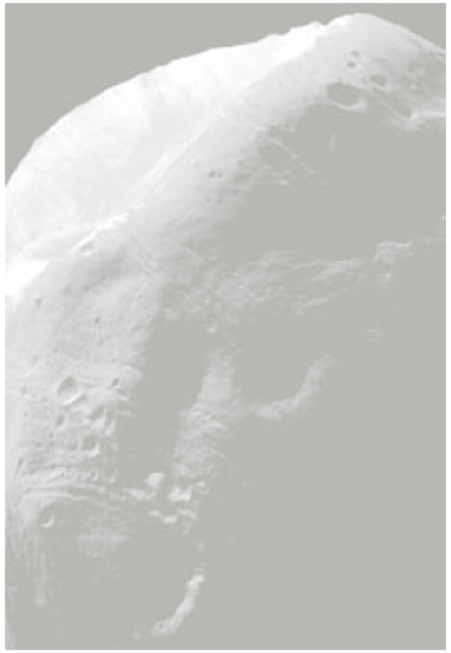


a b

FIGURE 3.23 (a) Image of the Mars moon Phobos taken by NASA's *Mars Global Surveyor.* (b) Histogram. (Original image courtesy of NASA.)

Example: Histogram Matching

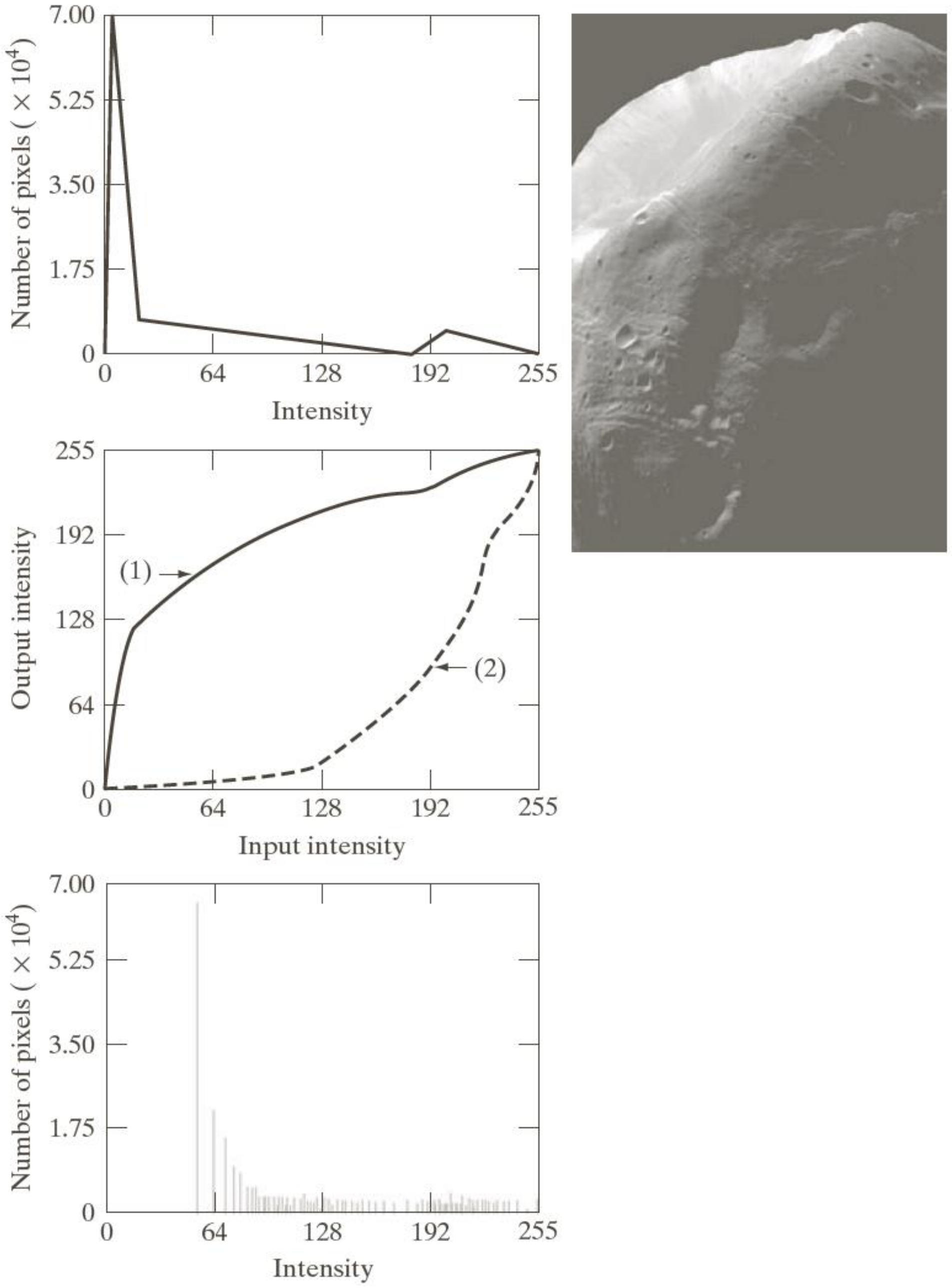




a b
c

FIGURE 3.24
(a) Transformation function for histogram equalization.
(b) Histogram-equalized image (note the washed-out appearance).
(c) Histogram of (b).

Example:



a c
b
d

FIGURE 3.25
(a) Specified histogram.
(b) Transformations.
(c) Enhanced image using mappings from curve (2).
(d) Histogram of (c).

Local Histogram Processing

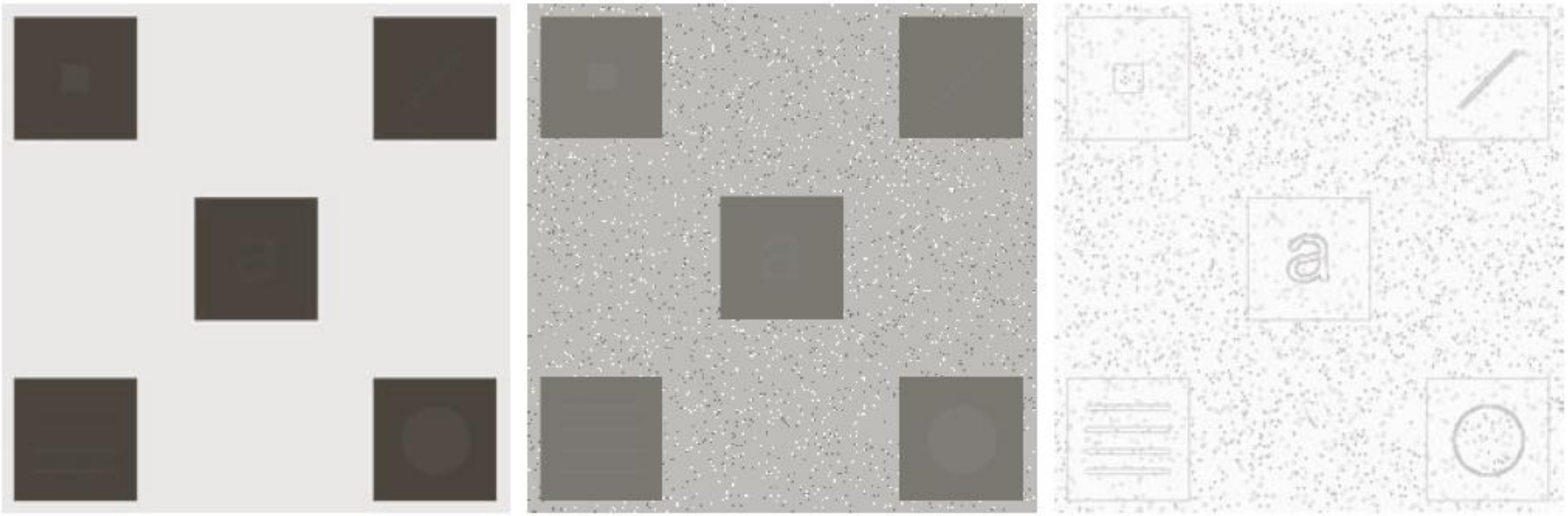
Define a neighborhood and move its center from pixel to pixel

At each location, the histogram of the points in the neighborhood is computed. Either histogram equalization or histogram specification transformation function is obtained

Map the intensity of the pixel centered in the neighborhood

Move to the next location and repeat the procedure

Local Histogram Processing: Example



a b c

FIGURE 3.26 (a) Original image. (b) Result of global histogram equalization. (c) Result of local histogram equalization applied to (a), using a neighborhood of size 3 × 3.

Using Histogram Statistics for Image Enhancement

Average Intensity

$$m = \sum_{i=0}^{L-1} r_i p(r_i) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)$$

$$u_n(r) = \sum_{i=0}^{L-1} (r_i - m)^n p(r_i)$$

Variance

$$\sigma^2 = u_2(r) = \sum_{i=0}^{L-1} (r_i - m)^2 p(r_i) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} [f(x, y) - m]^2$$

Using Histogram Statistics for Image Enhancement

Local average intensity

$$m_{s_{xy}} = \sum_{i=0}^{L-1} r_i \, p_{s_{xy}}(r_i)$$

s_{xy} denotes a neighborhood

Local variance

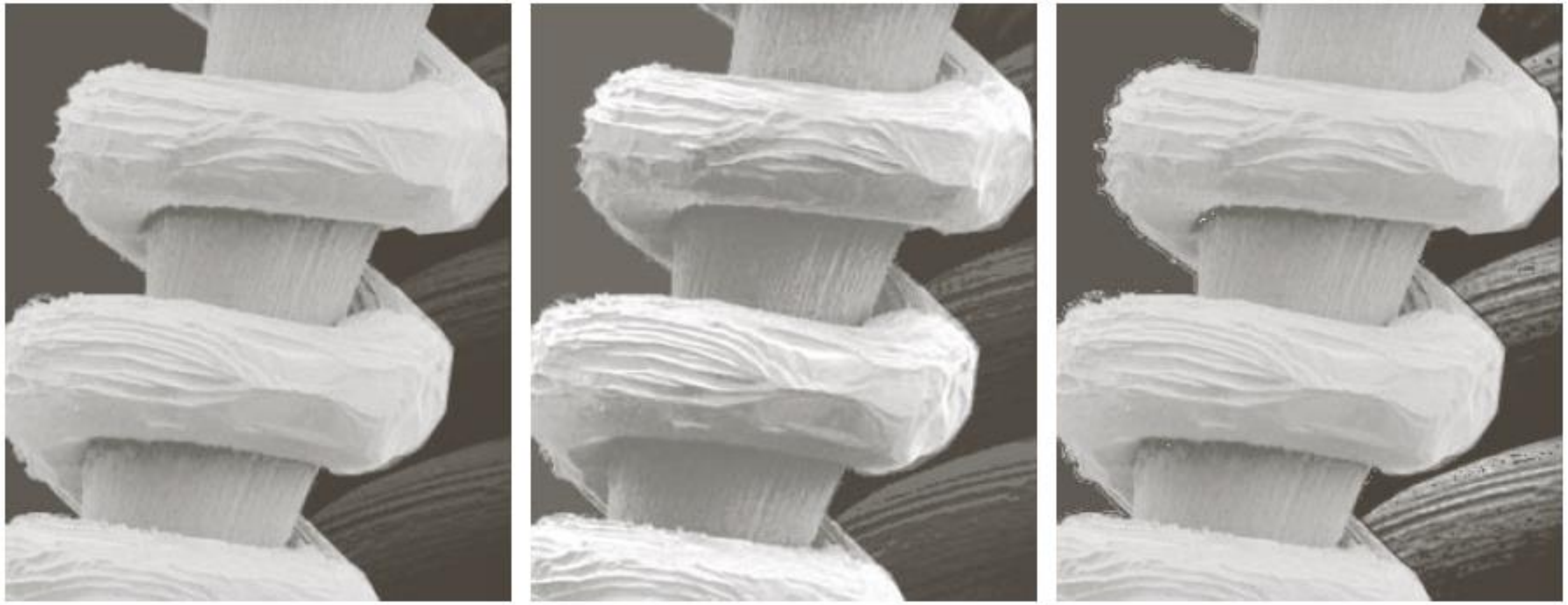
$$\sigma_{s_{xy}}^2 = \sum_{i=0}^{L-1} (r_i - m_{s_{xy}})^2 \, p_{s_{xy}}(r_i)$$

Using Histogram Statistics for Image Enhancement: Example

$g(x$

m_G

$k_0 =$



a b c

FIGURE 3.27 (a) SEM image of a tungsten filament magnified approximately 130×. (b) Result of global histogram equalization. (c) Image enhanced using local histogram statistics. (Original image courtesy of Mr. Michael Shaffer, Department of Geological Sciences, University of Oregon, Eugene.)