Filtering in the Frequency Domain

Outline

- Fourier Transform
- Filtering in Fourier Transform Domain

Fourier Series and Fourier Transform: History

- Fourier Series

  Any periodic function can be expressed as the sum of sines and /or cosines of different frequencies, each multiplied by a different coefficients

- Fourier Transform

  Any function that is not periodic can be expressed as the integral of sines and /or cosines multiplied by a weighing function

Fourier Series: Example

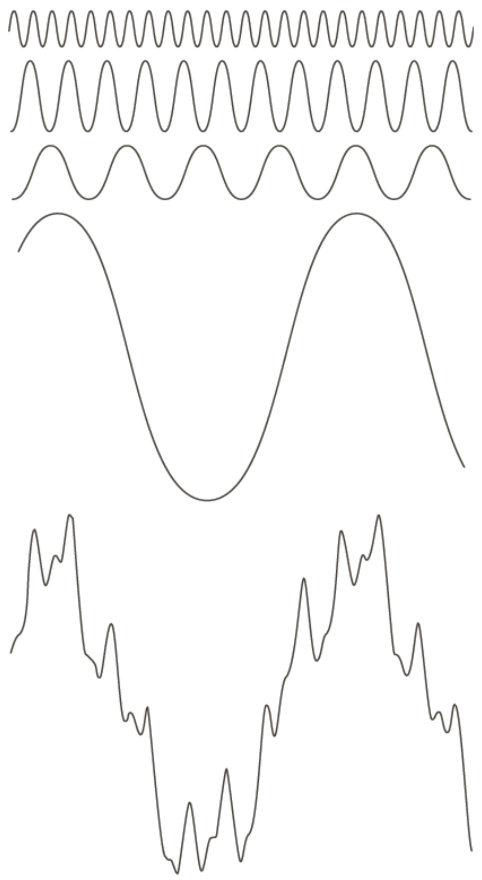


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

Preliminary Concepts

$$j = \sqrt{-1}, \text{ a complex number}$$

$$C = R + jI$$

the conjugate

$$C^* = R - jI$$

$$|C| = \sqrt{R^2 + I^2} \text{ and } \theta = \arctan(I / R)$$

$$C = |C|(\cos\theta + j\sin\theta)$$

Using Euler's formula,

$$C = |C| e^{j\theta}$$

Fourier Series

A function $f(t)$ of a continuous variable t that is periodic with period, T, can be expressed as the sum of sines and cosines multiplied by appropriate coefficients

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-j\frac{2\pi n}{T}t} dt \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots$$

Impulses and the Sifting Property (1)

A *unit impulse* of a continuous variable t located at t=0, denoted $\delta(t)$, defined as

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

and is constrained also to satisfy the identity

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

The *sifting property*

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

Impulses and the Sifting Property (2)

A *unit impulse* of a discrete variable x located at x=0, denoted $\delta(x)$, defined as

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

and is constrained also to satisfy the identity

$$\sum_{x=-\infty}^{\infty} \delta(x) = 1$$

The *sifting property*

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0)$$

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0)$$

Impulses and the Sifting Property (3)

$$\textit{impulse train } s_{\Delta T}(t),$$

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$

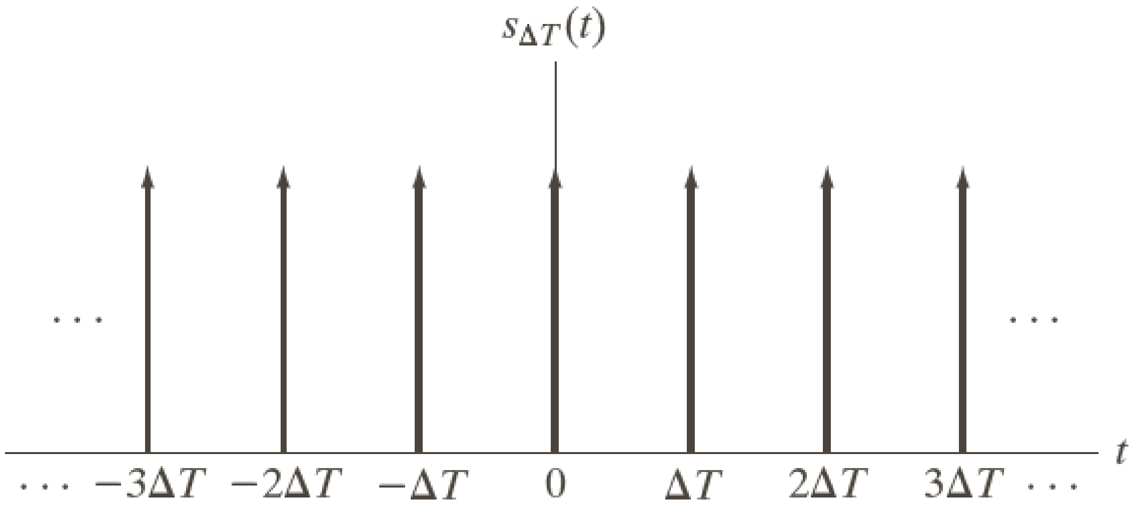


FIGURE 4.3 An impulse train.

Fourier Transform: One Continuous Variable

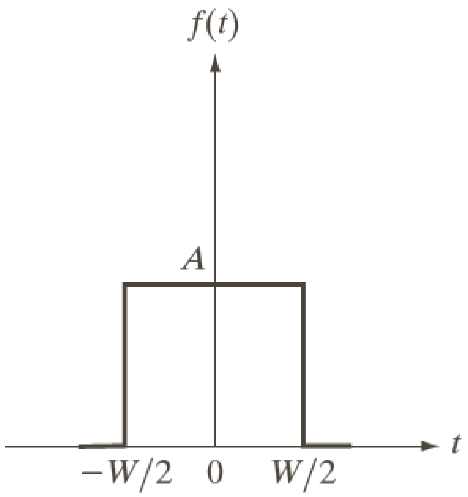
The *Fourier Transform* of a continous function $f(t)$

$$F(\mu) = \Im\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

The *Inverse Fourier Transform* of $F(\mu)$

$$f(t) = \Im^{-1}\{F(\mu)\} = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$

Fourier Transform: One Continuous Variable



a b c

FIGURE 4.4 (a) A simple func
infinity in both directions.

$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt = \int_{-W/2}^{W/2} A e^{-j2\pi\mu t} dt$$

$$= \frac{-A}{j2\pi\mu} \left[e^{-j2\pi\mu t} \right]_{-W/2}^{W/2} = \frac{A}{j2\pi W} \left[e^{j\pi\mu W} - e^{-j\pi\mu W} \right]$$

$$= AW \frac{\sin(\pi\mu W)}{(\pi\mu W)}$$

Fourier Transform: Impulses

The Fourier transform of a unit impulse located at the origin:

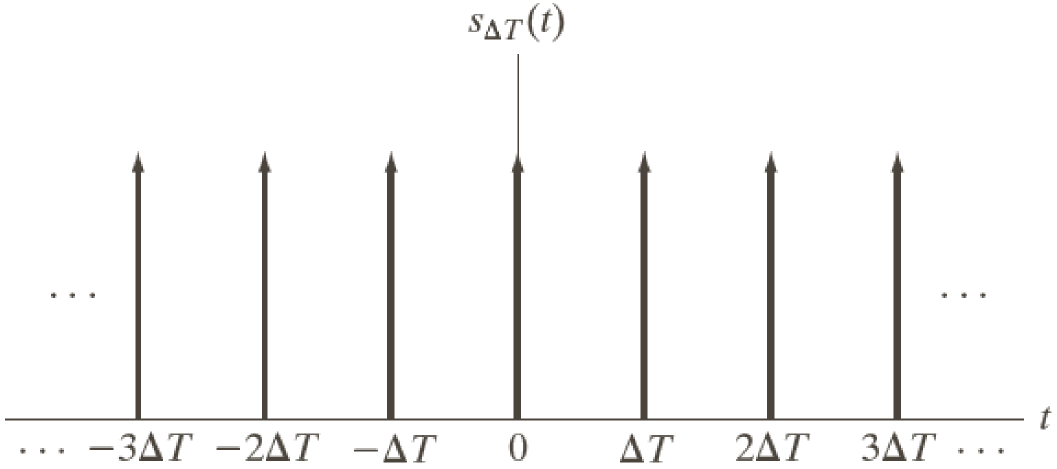
$$F(\mu) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi\mu t} dt$$
$$= e^{-j2\pi\mu 0}$$
$$= 1$$

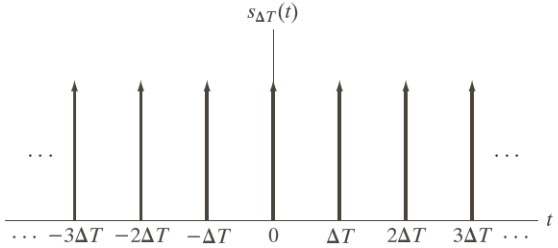
The Fourier transform of a unit impulse located at $t = t_0$:

$$F(\mu) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi\mu t} dt$$
$$= e^{-j2\pi\mu t_0}$$

Fourier Transform: Impulse Trains

Impulse train $s_{\Delta T}(t)$, $\quad s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$





The Fourier series:

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}t}$$

where

$$c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t) e^{-j\frac{2\pi n}{\Delta T}t} dt$$

Fourier Transform: Impulse Trains

$$c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t) e^{-j\frac{2\pi n}{\Delta T}t} dt = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} \delta(t) e^{-j\frac{2\pi n}{\Delta T}t} dt$$

$$= \frac{1}{\Delta T} e^{0} = \frac{1}{\Delta T}$$

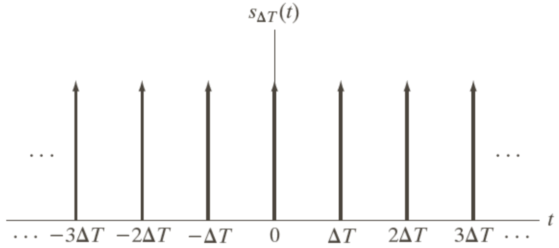
$$\Im\left\{ e^{j\frac{2\pi n}{\Delta T}t} \right\} = \int_{-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t} e^{-j2\pi\mu t} dt = \int_{-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t} e^{-j2\pi\mu t} dt$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi(\mu - \frac{n}{\Delta T})t} dt = \delta(\mu - \frac{n}{\Delta T})$$

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}t} \qquad = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}$$

$$\Im^{-1}\left\{ \delta(\mu - \frac{n}{\Delta T}) \right\} = \int_{-\infty}^{\infty} \delta(\mu - \frac{n}{\Delta T}) e^{j2\pi\mu t} du$$

$$= e^{j\frac{2\pi n t}{\Delta T}}$$



Fourier Transform: Impulse Trains

Let $S(\mu)$ denote the Fourier transform of the periodic impulse train $S_{\Delta T}(t)$

$$S(\mu) = \Im\{S_{\Delta T}(t)\} = \Im\left\{\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\}$$

$$= \frac{1}{\Delta T}\Im\left\{\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\}$$

$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \delta(\mu - \frac{n}{\Delta T})$$

Fourier Transform and Convolution

The convolution of two functions is denoted by the operator $\star$

$$f(t) \star h(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau$$

Fourier Transform and Convolution

Fourier Transform Pairs

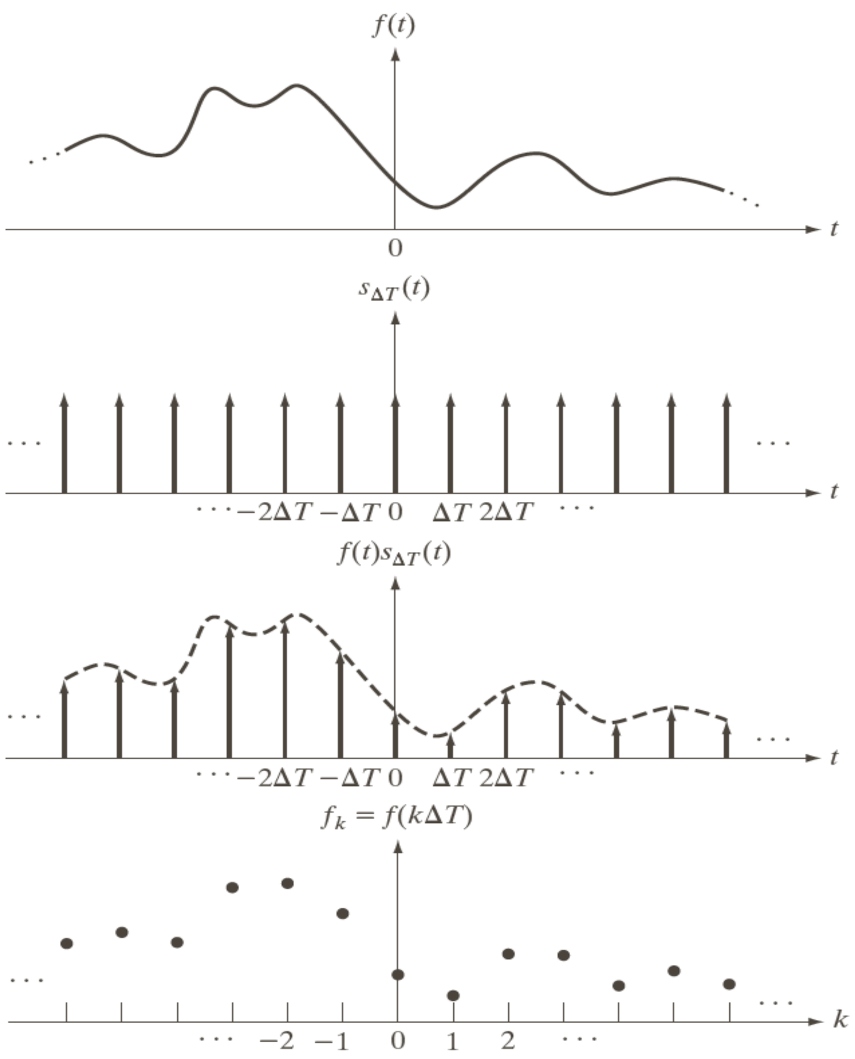
$$f(t) \star h(t) \Leftrightarrow H(\mu)F(\mu)$$

$$f(t)h(t) \Leftrightarrow H(\mu) \star F(\mu)$$

Fourier Transform of Sampled Functions

$$\tilde{f}(t) = f(t)s_{\Delta T}(t)$$

$$= \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$



a
b
c
d

FIGURE 4.5
(a) A continuous function. (b) Train of impulses used to model the sampling process. (c) Sampled function formed as the product of (a) and (b). (d) Sample values obtained by integration and using the sifting property of the impulse. (The dashed line in (c) is shown for reference. It is not part of the data.)

Fourier Transform of Sampled Functions

$$\tilde{F}(\mu) = \Im\{\tilde{f}(t)\} = \Im\{f(t)s_{\Delta T}(t)\} = F(\mu) \quad S(\mu)$$

?

$$\star S(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta(\mu - \frac{n}{\Delta T})$$

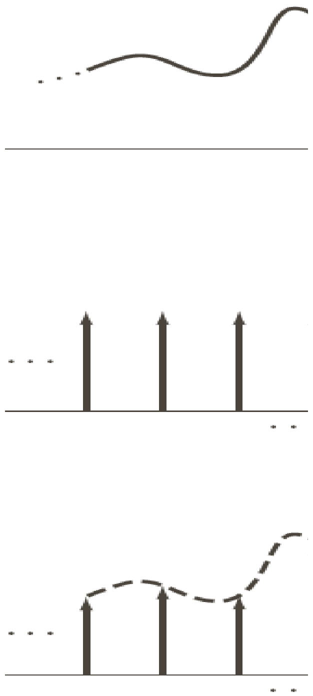
$$\tilde{F}(\mu) = F(\mu) \quad S(\mu) = \int_{-\infty}^{\infty} F(\tau)S(\mu-\tau)d\tau$$

$$= \frac{1}{\Delta T}\int_{-\infty}^{\infty} F(\tau)\sum_{n=-\infty}^{\infty}\delta(\mu-\tau-\frac{n}{\Delta T})d\tau$$

$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} F(\tau)\delta(\mu-\tau-\frac{n}{\Delta T})d\tau$$

$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F(\mu-\frac{n}{\Delta T})$$

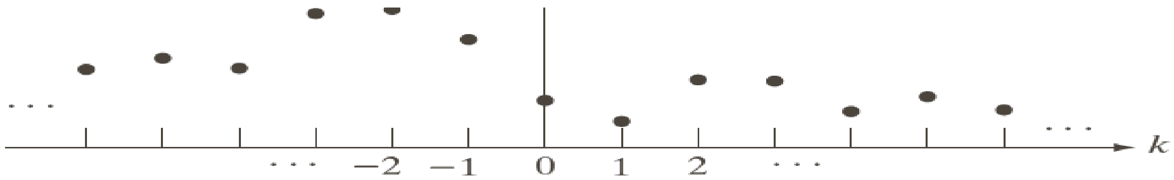
Question



The Fourier transform of the sampled function (shown in the following figure) is

1. Continuous

2. Discrete



Fourier Transform of Sampled Functions

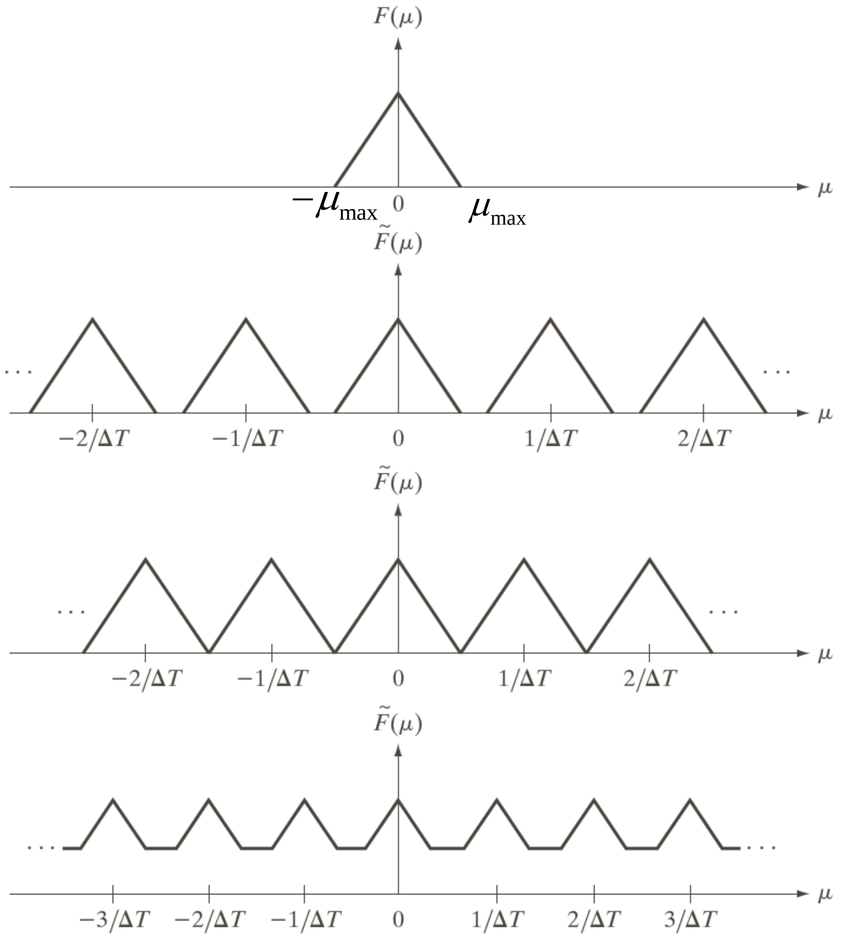
- A **bandlimited** signal is a signal whose Fourier transform is zero above a certain finite frequency. In other words, if the Fourier transform has finite support then the signal is said to be bandlimited.

An example of a simple bandlimited signal is a sinusoid of the form,

$$x(t) = \sin(2\pi ft + \theta)$$

Fourier Transform of Sampled Functions

$$\tilde{F}(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$



Over-sampling

$$\frac{1}{\Delta T} > 2\mu_{max}$$

Critically-sampling

$$\frac{1}{\Delta T} = 2\mu_{max}$$

under-sampling

$$\frac{1}{\Delta T} < 2\mu_{max}$$

Nyquist–Shannon sampling theorem

- We can recover $f(t)$ from its sampled version if we can isolate a copy of $F(\mu)$ from the periodic sequence of copies of this function contained in $\tilde{F}(\mu)$, the transform of the sampled function $\tilde{f}(t)$
- Sufficient separation is guaranteed if $\frac{1}{\Delta T} > 2\mu_{\max}$

Sampling theorem: A continuous, band-limited function can be recovered completely from a set of its samples if the samples are acquired at a rate exceeding twice the highest frequency content of the function

Nyquist–Shannon sampling theorem

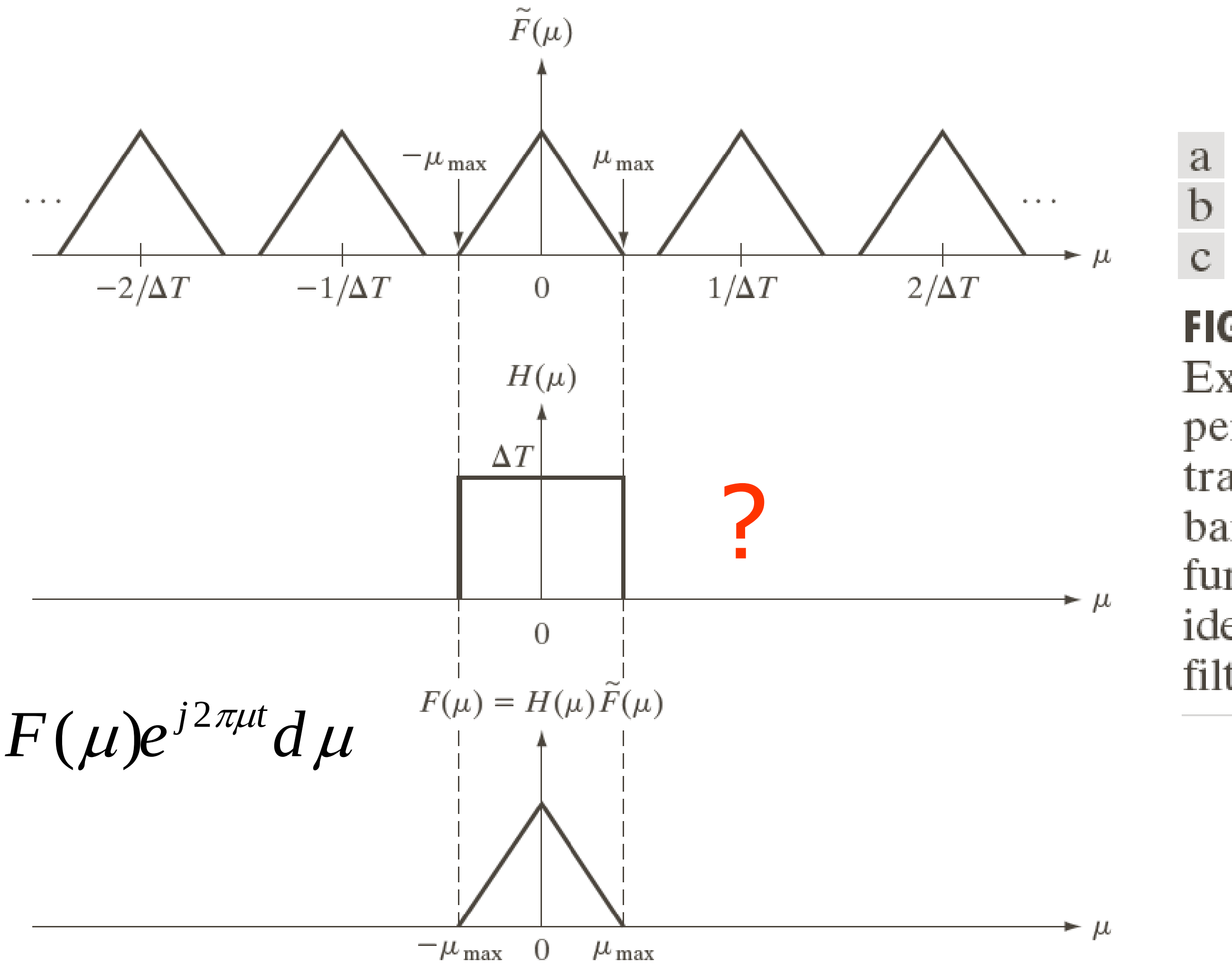


FIGURE 4.8 Extracting one period of the transform of a band-limited function using an ideal lowpass filter.

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$

Aliasing

If a band-limited function is sampled at a rate that is less than twice its highest frequency?

The inverse transform will yield a corrupted function. This effect is known as ***frequency aliasing*** or simply as ***aliasing***.

Aliasing

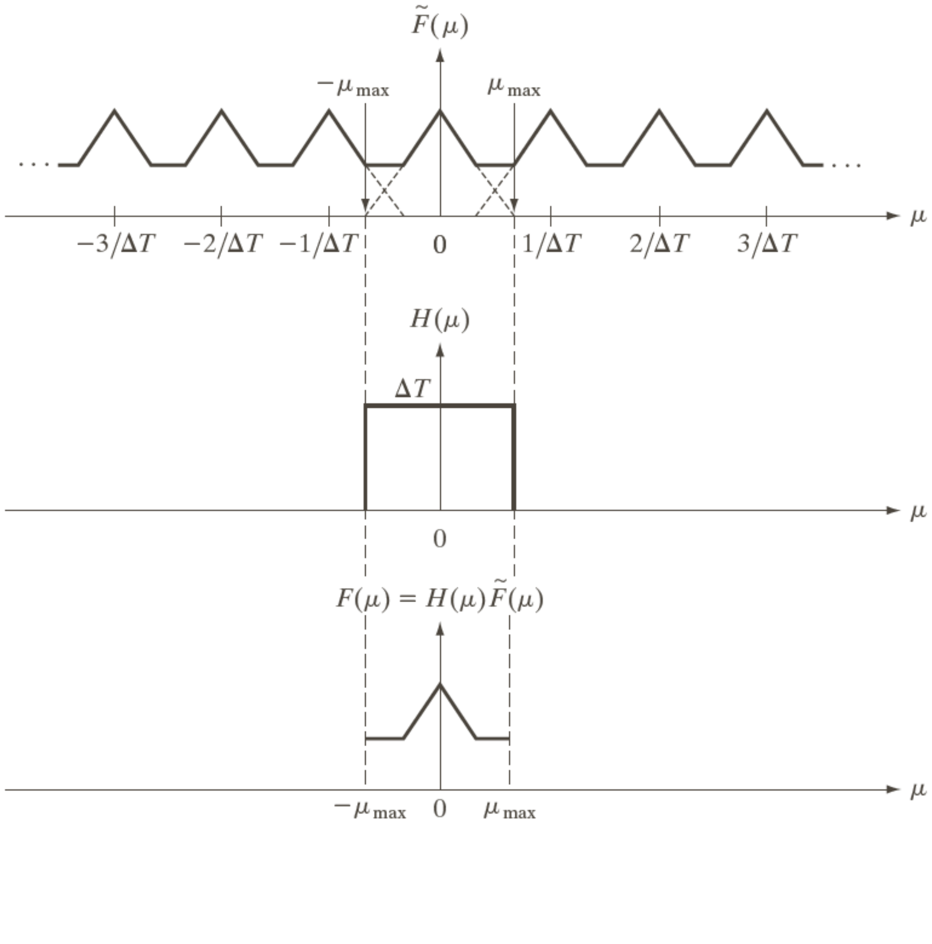


FIGURE 4.9 (a) Fourier transform of an under-sampled, band-limited function. (Interference from adjacent periods is shown dashed in this figure). (b) The same ideal lowpass filter used in Fig. 4.8(b). (c) The product of (a) and (b). The interference from adjacent periods results in aliasing that prevents perfect recovery of $F(\mu)$ and, therefore, of the original, band-limited continuous function. Compare with Fig. 4.8.

Aliasing

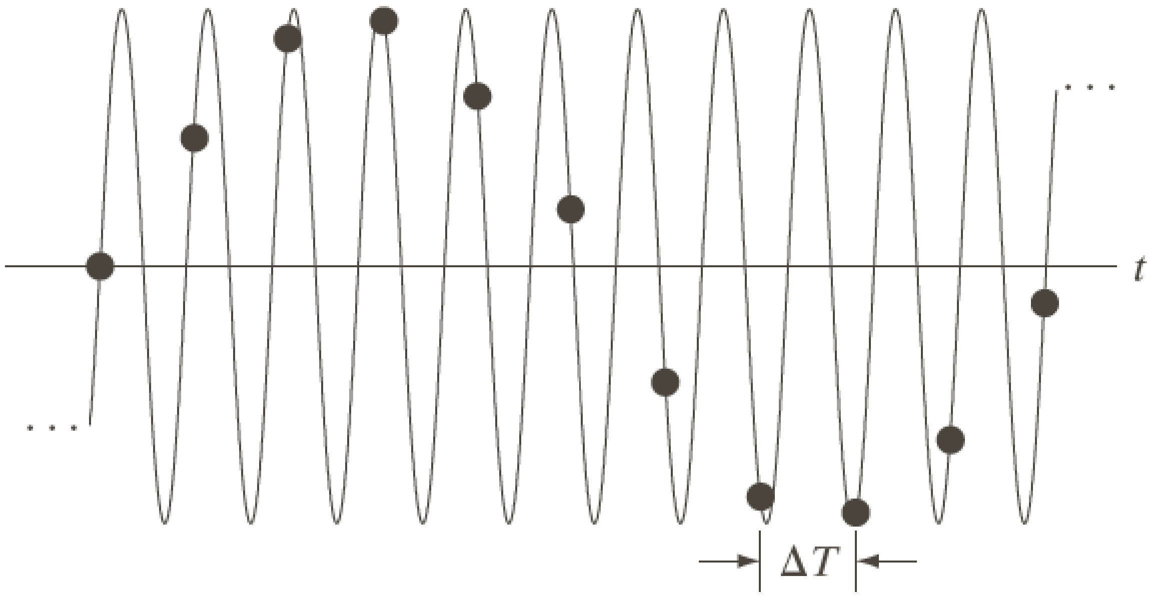


FIGURE 4.10 Illustration of aliasing. The under-sampled function (black dots) looks like a sine wave having a frequency much lower than the frequency of the continuous signal. The period of the sine wave is 2 s, so the zero crossings of the horizontal axis occur every second. ΔT is the separation between samples.

Function Reconstruction from Sampled Data

$$F(\mu) = H(\mu)\tilde{F}(\mu)$$

$$f(t) = \mathfrak{I}^{-1}\{F(\mu)\}$$

$$= \mathfrak{I}^{-1}\{H(\mu)\tilde{F}(\mu)\}$$

$$= h(t) \star \tilde{f}(t)$$

$$f(t) = \sum_{n=-\infty}^{\infty} f(n\Delta T)\operatorname{sinc}\left[(t - n\Delta T)/n\Delta T\right]$$

The Discrete Fourier Transform (DFT) of One Variable

$$F(\mu)=\sum_{x=0}^{M-1} f(x)e^{-j2\pi\mu x/M}, \qquad \mu=0,1,\ldots,M-1$$

$$f(x)=\frac{1}{M}\sum_{\mu=0}^{M-1} F(\mu)e^{j2\pi\mu x/M}, \qquad x=0,1,2,\ldots,M-1$$

2-D Impulse and Sifting Property: Continuous

The impulse $\delta(t, z)$, $\quad \delta(t,z) = \begin{cases} \infty & \text{if } t = z = 0 \\ 0 & \text{otherwise} \end{cases}$

and $\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t,z)\,dt\,dz = 1$

The sifting property

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\delta(t,z)\,dt\,dz = f(0,0)$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\delta(t-t_0, z-z_0)\,dt\,dz = f(t_0, z_0)$$

2-D Impulse and Sifting Property: Discrete

The impulse $\delta(x, y)$, $$\delta(x,y) = \begin{cases} 1 & \text{if } x = y = 0 \\ 0 & \text{otherwise} \end{cases}$$

The sifting property

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x,y)\delta(x,y) = f(0,0)$$

and

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x,y)\delta(x-x_0, y-y_0) = f(x_0, y_0)$$

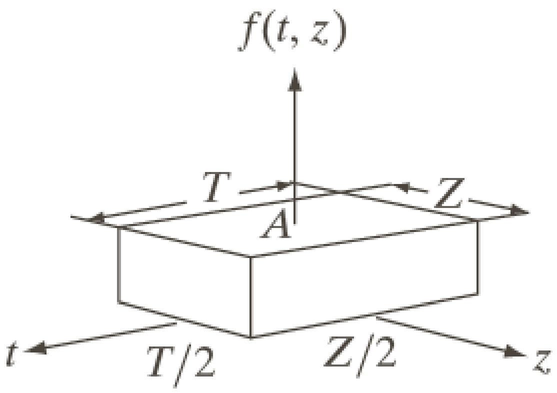
2-D Fourier Transform: Continuous

$$F(\mu,\nu)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(t,z)e^{-j2\pi(\mu t+\nu z)}dtdz$$

and

$$f(t,z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\mu,\nu)e^{j2\pi(\mu t+\nu z)}d\mu d\nu$$

2-D Fourier Transform: Continuous



$$F(\mu,\nu)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(\mu t+\nu z)}dtdz$$

$$=\int_{-T/2}^{T/2}\int_{-Z/2}^{Z/2} Ae^{-j2\pi(\mu t+\nu z)}dtdz$$

$$=ATZ\left[\frac{\sin(\pi\mu T)}{\pi\mu T}\right]\left[\frac{\sin(\pi\nu T)}{\pi\nu T}\right]$$

a b

FIGURE 4.13 (a) A 2-D function, and (b) a section of its spectrum (not to scale). The block is longer along the t-axis, so the spectrum is more "contracted" along the μ-axis. Compare with Fig. 4.4.

2-D Sampling and 2-D Sampling Theorem

$2-D$ impulse train:

$$s_{\Delta T \Delta Z}(t,z)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(t-m\Delta T, z-n\Delta Z)$$

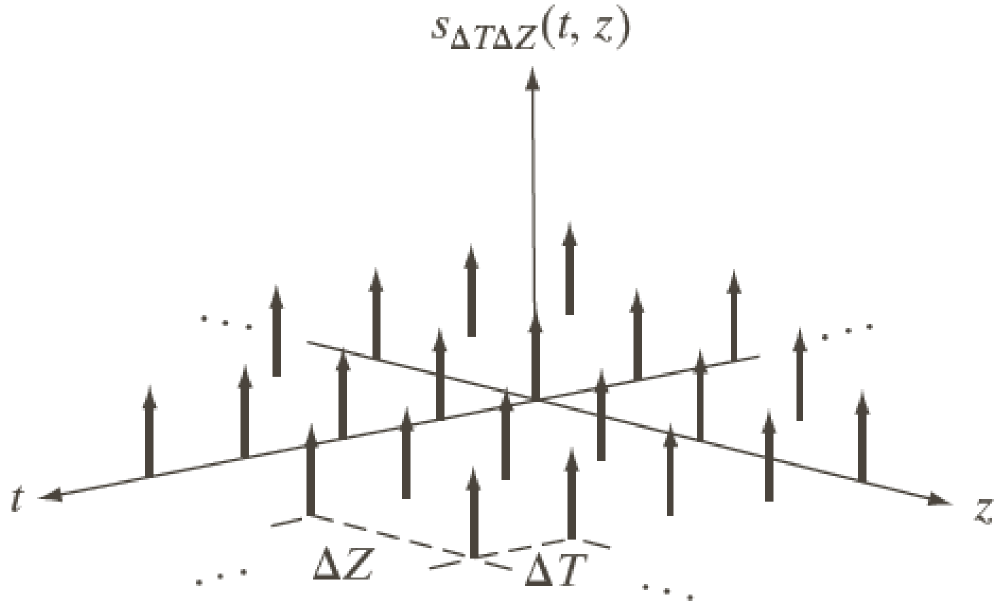


FIGURE 4.14 Two-dimensional impulse train.

2-D Sampling and 2-D Sampling Theorem

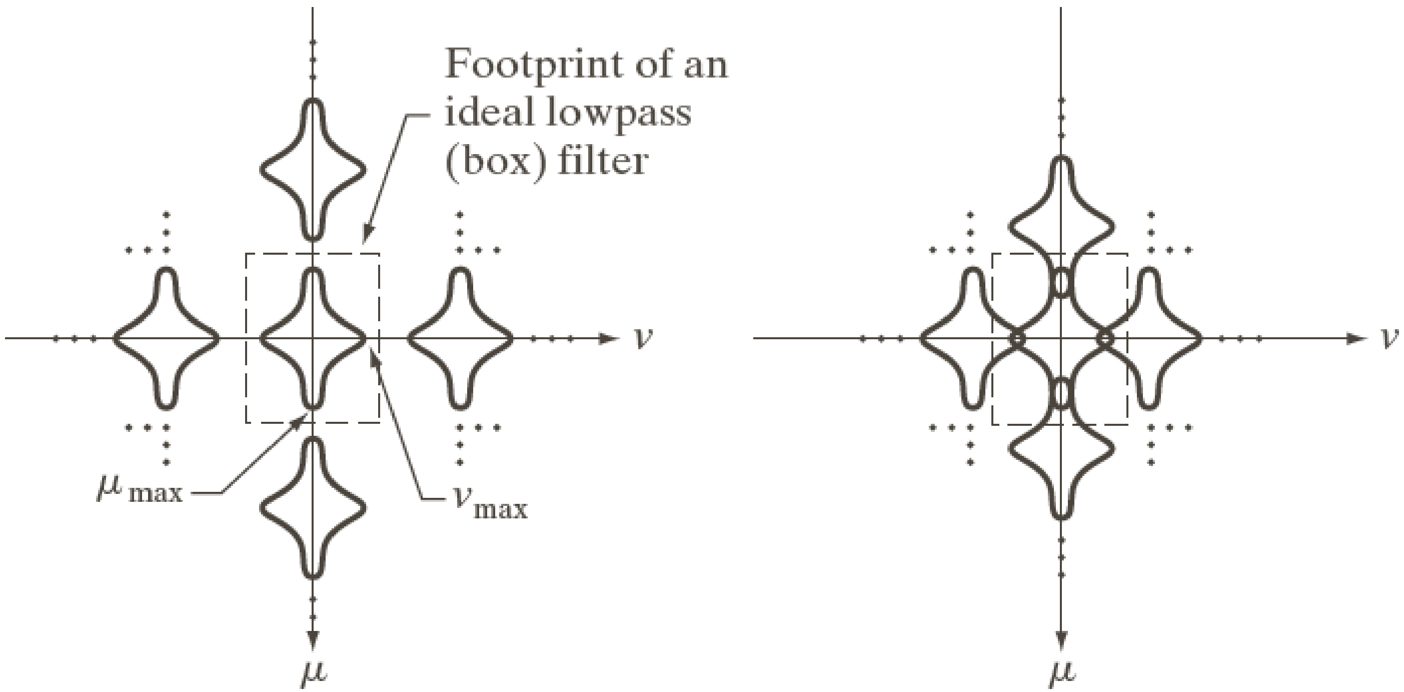
Function $f(t,z)$ is said to be band-limited if its Fourier transform is 0 outside a rectangle established by the intervals $[-\mu_{max}, \mu_{max}]$ and $[-\nu_{max}, \nu_{max}]$, that is

$$F(\mu,\nu) = 0 \text{ for } |\mu| \geq \mu_{max} \text{ and } |\nu| \geq \nu_{max}$$

Two-dimensional sampling theorem:
A continuous, band-limited function $f(t,z)$ can be recovered with no error from a set of its samples if the sampling intervals are

$$\Delta T < \frac{1}{2\mu_{max}} \text{ and } \Delta Z < \frac{1}{2\nu_{max}}$$

2-D Sampling and 2-D Sampling Theorem



a b

FIGURE 4.15 Two-dimensional Fourier transforms of (a) an over-sampled, and (b) under-sampled band-limited function.

Aliasing in Images: Example

In an image system, the number of samples is fixed at 96x96 pixels. If we use this system to digitize checkerboard patterns …

Under-sampling

a	b
c	d

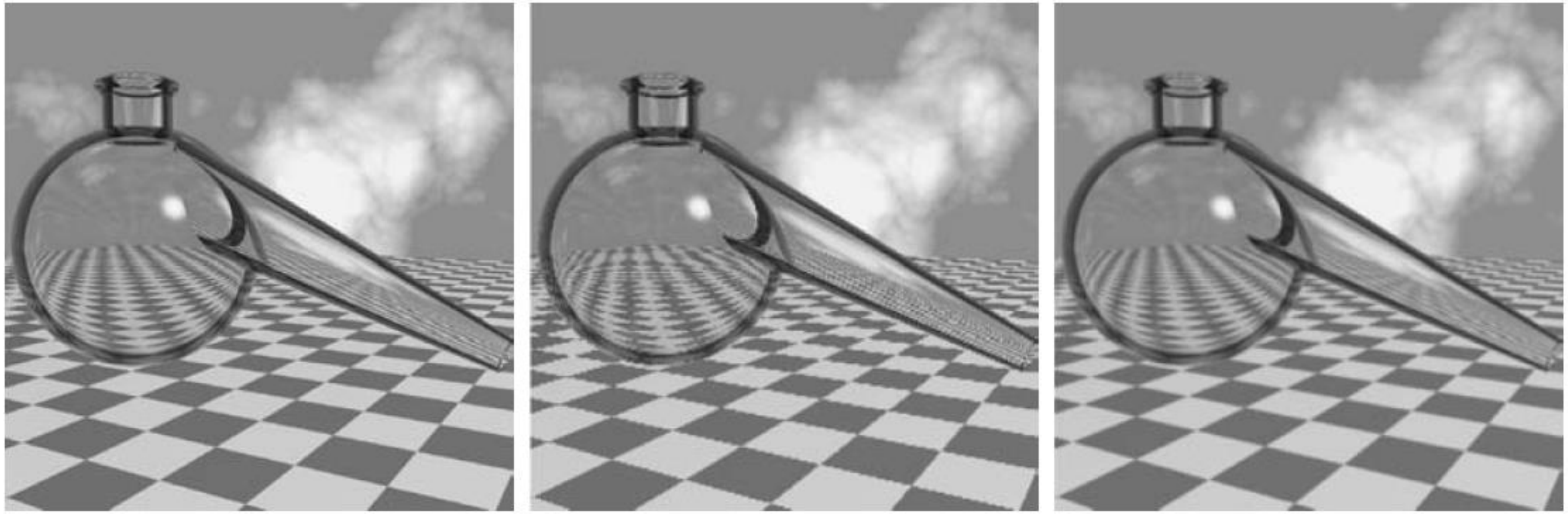
FIGURE 4.16 Aliasing in images. In (a) and (b), the lengths of the sides of the squares are 16 and 6 pixels, respectively, and aliasing is visually negligible. In (c) and (d), the sides of the squares are 0.9174 and 0.4798 pixels, respectively, and the results show significant aliasing. Note that (d) masquerades as a "normal" image.

Aliasing in Images: Example

a b c

FIGURE 4.17 Illustration of aliasing on resampled images. (a) A digital image with negligible visual aliasing. (b) Result of resizing the image to 50% of its original size by pixel deletion. Aliasing is clearly visible. (c) Result of blurring the image in (a) with a 3×3 averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)

Aliasing in Images: Example



a b c

Re-sampling

FIGURE 4.18 Illustration of jaggies. (a) A 1024 × 1024 digital image of a computer-generated scene with negligible visible aliasing. (b) Result of reducing (a) to 25% of its original size using bilinear interpolation. (c) Result of blurring the image in (a) with a 5 × 5 averaging filter prior to resizing it to 25% using bilinear interpolation. (Original image courtesy of D. P. Mitchell, Mental Landscape, LLC.)

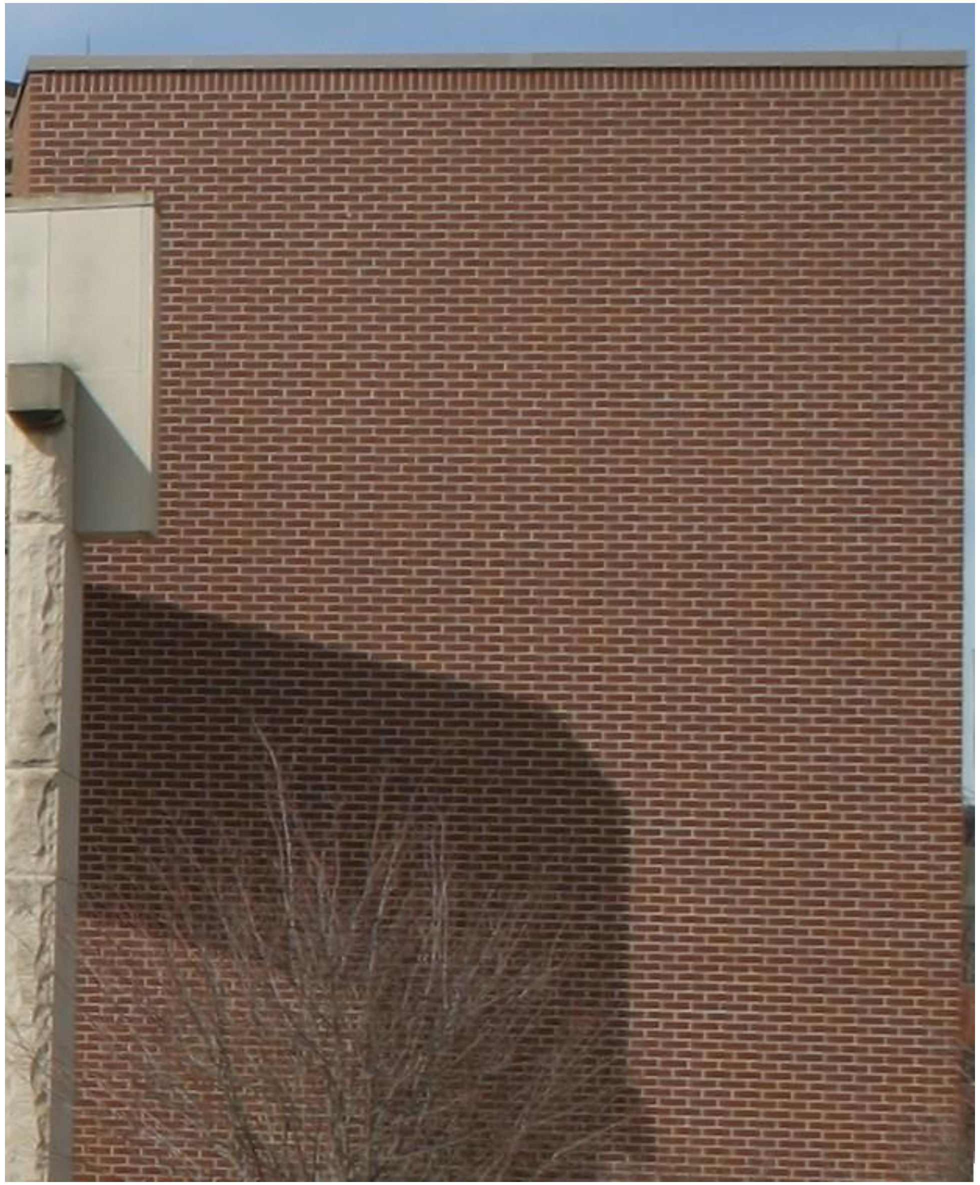
Moiré patterns

- Moiré patterns are often an undesired artifact of images produced by various digital imaging and computer graphics techniques

  e. g., when scanning a halftone picture or ray tracing a checkered plane. This cause of moiré is a special case of aliasing, due to under-sampling a fine regular pattern

http://en.wikipedia.org/wiki/Moiré_pattern

Moiré patterns



Moiré patterns

A moiré pattern formed by incorrectly down-sampling the former image

2-D Discrete Fourier Transform and Its Inverse

DFT:

$$F(\mu,\nu)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(\mu x/M+\nu y/N)}$$

$$\mu=0,1,2,\ldots,M-1;\nu=0,1,2,\ldots,N-1;$$

$f(x,y)$ is a digital image of size $\mathrm{M}\times\mathrm{N}$.

IDFT:

$$f(x,y)=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} F(\mu,\nu)e^{j2\pi(\mu x/M+\nu y/N)}$$

Properties of the 2-D DFT

relationships between spatial and frequency intervals

Let ΔT and ΔZ denote the separations between samples, then the seperations between the corresponding discrete, frequency domain variables are given by

$$\Delta\mu = \frac{1}{M\Delta T}$$

and $$\Delta v = \frac{1}{N\Delta Z}$$

Properties of the 2-D DFT
translation and rotation

$$f(x,y)e^{j2\pi(\mu_0 x/M+\nu_0 y/N)} \Leftrightarrow F(\mu-\mu_0,\nu-\nu_0)$$

and

$$f(x-x_0,y-y_0) \Leftrightarrow F(\mu,\nu)e^{-j2\pi(\mu x_0/M+\nu y_0/N)}$$

Using the polar coordinates

$$x = r\cos\theta \quad y = r\sin\theta \quad \mu = \omega\cos\varphi \quad \nu = \omega\sin\varphi$$

results in the following transform pair:

$$f(r,\theta+\theta_0) \Leftrightarrow F(\omega,\varphi+\theta_0)$$

Properties of the 2-D DFT

periodicity

$2-D$ Fourier transform and its inverse are infinitely periodic

$$F(\mu,\nu)=F(\mu+k_1M,\nu)=F(\mu,\nu+k_2N)=F(\mu+k_1M,\nu+k_2N)$$

$$f(x,y)=f(x+k_1M,y)=f(x,y+k_2N)=f(x+k_1M,y+k_2N)$$

$$f(x)e^{j2\pi(\mu_0 x/M)}\Leftrightarrow F(\mu-\mu_0)$$

$$\mu_0=M/2,\quad f(x)(-1)^x\Leftrightarrow F(\mu-M/2)$$

$$f(x,y)(-1)^{x+y}\Leftrightarrow F(\mu-M/2,\nu-N/2)$$

Properties of the 2-D DFT

periodicity

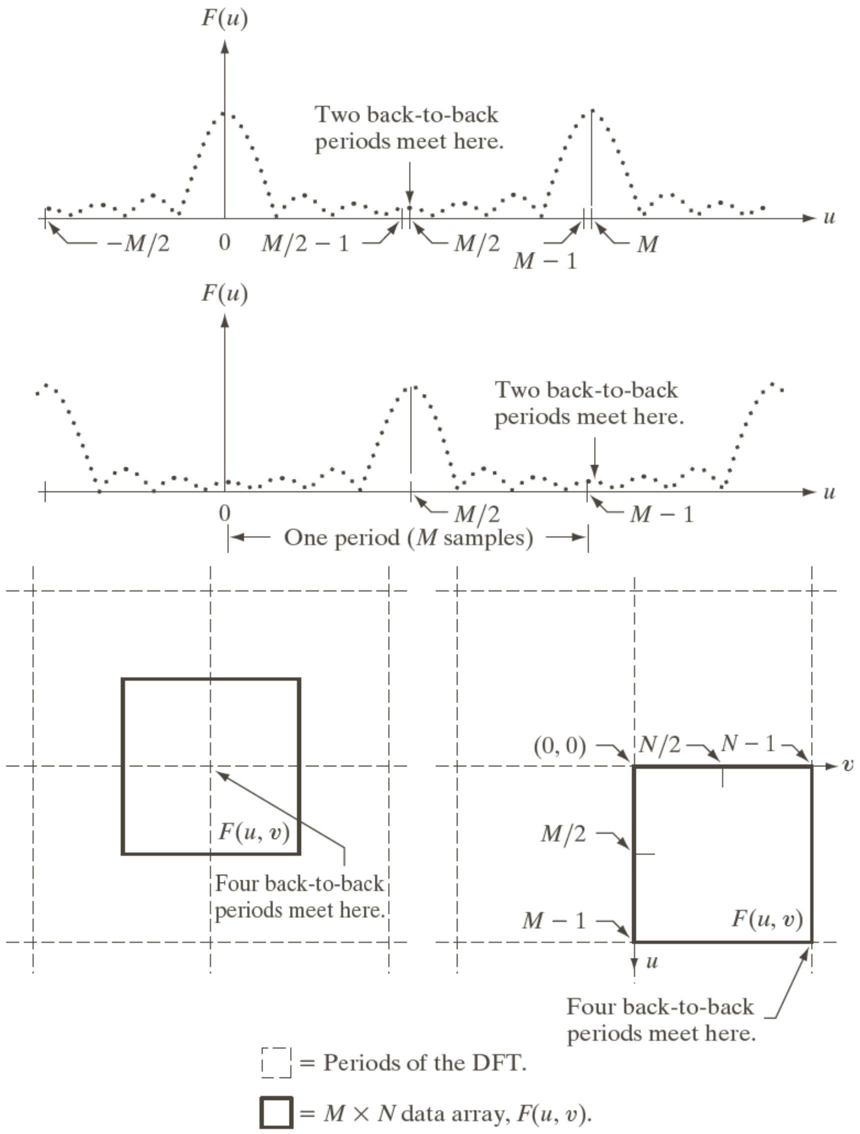


FIGURE 4.23 Centering the Fourier transform. (a) A 1-D DFT showing an infinite number of periods. (b) Shifted DFT obtained by multiplying $f(x)$ by $(-1)^x$ before computing $F(u)$. (c) A 2-D DFT showing an infinite number of periods. The solid area is the $M \times N$ data array, $F(u,v)$, obtained with Eq. (4.5-15). This array consists of four quarter periods. (d) A Shifted DFT obtained by multiplying $f(x,y)$ by $(-1)^{x+y}$ before computing $F(u,v)$. The data now contains one complete, centered period, as in (b).